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# AN EXTENSION OF INFINITESIMAL CALCULUS

By C. Schmieden and D. Laugwitz

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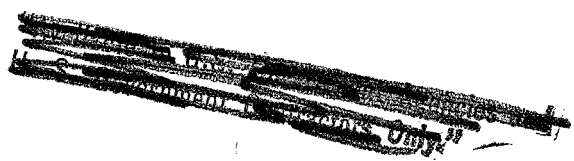

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# AN EXTENSION OF INFINITESIMAL CALCULUS

By

Curt Schmieden and Detlef Laugwitz

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Introduction. The concept of Cantor-Dedekind's real numbers and thus all of today's analysis rest on the agreement that two sequences of rational numbers assume the same limiting value, if their members finally differ by less than $\varepsilon$, where $\varepsilon$ is an arbitrary, positive, rational number. Equivalent to this is Dedekind's conception: Every "section" in the domain of the rational numbers is generated by precisely one real number.

One will note that this agreement is subject to a certain arbitrariness. Let us consider a section, that is a division of the rational numbers into two classes, such that each number of the first class is smaller than each member of the second class. Then, as is well known, from the "continuity" of the number scale Dedekind infers the postulate of the existence of a "real" number dividing the two classes. That there is only one dividing number in each "section", is another assumption not proven by Dedekind.

In the course of this article we shall prove, at least in principle, that a non-Dedekindian analysis is indeed possible, an analysis on the basis of a number concept, for which Dedekind's postulate (each section corresponds to just one real number) is not valid; in that case, of course, the equivalent formulation of Cantor's is not fulfilled either. In consequence, we permit two sequences of rational numbers, which, in conventional analysis, tend to the same real limiting value, to have different limiting values. The most radical possibility is to postulate that the limiting value of two different sequences is always different; this assumption we shall use in the present paper.

In case one drops Cantor's definition of the equivalence of number-sequences, it would seem unreasonable to still uphold Cauchy's convergence principle, which is similar in its formulation and serves to select the "convergent" sequences out of all possible sequences. One could rather try to introduce limiting values also for as of now divergent sequences. Again we want to use the most radical assumption, namely the assignment of a limiting value to any arbitrary sequence of rational numbers.

In Chapter I we shall investigate the method of calculating with these limiting values. The elementary operations of calculation and the "larger than"-relation can be explained in the familiar manner. It becomes apparent that the domain of numbers contains infinitely large numbers, that is numbers which are larger than any natural number. Correspondingly there are also infinitesimally small positive numbers. The number domain appears to be a ring, but contains zero-divisors and is not totally arranged (because of the "larger"-relation.) These two facts, at first sight seemingly disturbing, are brought about by admitting arbitrary sequences of rational numbers in the definition of our numbers. It could be attempted to arrive at a totally ordered body by imposing restrictions on the admissible sequences. For two reasons this is not advisable. For once the zero-divisor and the non-comparability of some pairs of elements do not cause any difficulty whatsoever; added restrictions on the defining sequences would rather complicate the proofs. And secondly it is very

appropriate if a limiting value is defined for each sequence, because this eliminates some divergence difficulties.

In chapter II the foundations of analysis are being developed, which in the present paper leads up to the principal theorems of differential and integral calculus, adequate for presentation in the beginning terms of a mathematics student. In Chapter III we shall discuss some special examples and applications.

It should be noted that herewith the usual analysis has not just been rebuilt or modified, but rather has been extended in a genuine way. From which respects this is a valid opinion we shall see later. At this point we just like to point out, that in Chapter III, Par. 2 we shall discuss functions with the properties of Dirac's $\delta$-function. These functions can be introduced here without extension of the concept of function and without added new concepts at all, since the notions of continuity, differentiation and integration are also applicable in the case of functions performing infinitely large jumps within infinitesimally small intervals. Also, the multiplication of functions of this kind - much opposed to the conditions encountered in distribution theory - does not meet difficulties.

Other points of view worthwhile mentioning are the complete interchangeability of the limiting value processes and the possibility of arbitrarily rearranging all infinite series (regardless of whether, in the conventional manner of thinking, the series are merely conditionally convergent.) We refer the reader to Chapter I, Par. 5 and Chapter III, Par. 4.

Some historical remarks should be in place here. Ever since the development of mathematics converged towards the invention of infinitesimal calculus, infinitesimally small numbers held a prominant role in the discussions of the mathematicians. Even after successfully bypassing the difficulties presented by the "infinitesimal" during the 19th century again and again attempts have been made to arrive at a satisfying concept of the "infinitesimally small". One should particularly mention the considerations of P. Du Bois-Reymond. A new impuls was given to the matter by Veronese's book "Foundations of Geometry", in which he investigated non-archimedian comparison of paths. Basing on this work T. Levi-Civita and H. Hahn developed non-archimedian systems of quantities.[1] Furthermore we all know of Hilbert's example of a non-archimedian geometry. K. Geissler launched an inexact attempt of explaining the "infinitesimal" in the analysis.[2] While it was not attempted to establish their relation to analysis, non-archimedian bodies have widely been investigated since the work of Artin, Schreier, and Baer.[3] At first sight the establishment of such a relation is seemingly excluded, because analysis requires rather concrete working concepts, that is concepts which in a specific case can be established in terms of the natural numbers. As the final conclusion in regard to the historical development of the problem of the arithmetical fixation of the lone state in the continuum H. Weyl[4] announced the following principle: The various natural numbers are the object of the number theory, the possible magnitudes (or infinite sequences) of natural

numbers are the object of the theory of the continuum.

[1]) Bois-Reymond, P. du: Die allgemeine Funktionentheorie. I. Tübingen 1882. — Veronese, G.: Fondamenti di geometria, Padua 1891; deutsch: Grundzüge der Geometrie. Leipzig 1894. — Levi-Civita, T.: Sugli infiniti ed infinitesimi attuali. Atti R. Istituto Veneto (7) 4, 1765—1815 (1905). — Hahn, H.: Über die nicht-archimedischen Größensysteme. Wiener Ber. 116, 601—655 (1907). — Vorher hat sich schon B. Bolzano, Paradoxien des Unendlichen, Leipzig 1851, (auch Philos. Bibl. Bd. 99) mit der Problematik des Rechnens mit unendlich großen und unendlich kleinen Zahlen auseinandergesetzt.

[2]) J.-Ber. DMV. 13 (1904); daselbst auch eine Diskussion mit F. Bernstein u. a.

[3]) Artin, E., u. O. Schreier: Algebraische Konstruktion reeller Körper. Hamb. Abh. 5. 85—99 (1926). = Baer, R.: Über nicht-archimedisch geordnete Körper. Sitzgsber. Heidelb. 8. Abh., 3—13 (1927).

[4]) Weyl, H.: Philosophie der Mathematik und Naturwissenschaft, p. 33. 1928.

Clarified by an example H. Weyl is saying: It is not sufficient to adjunct infinitely large numbers into the field of rational numbers, such that one can perform the rational operations with the elements of the newly formed domain. But rather, in addition to this, it must be established, in which manner (perhaps as the limiting value of one particular sequence) each element can be generated from the rational numbers. Furthermore the numbers have to be such that an analysis can be constructed from them. Thus, for instance, it must be required that infinitely small numbers can be summed up to finite numbers. In other words, sums are needed in which one or both limits constitute infinitely large "whole" numbers.

For readers wishing to skip over to Chapter III (examples of applications) a brief nomenclature is given following Chapter I. Aside from occasional back-references it suffices to read the first two paragraphs of Chapter I in order to gain access to the following two chapters.

## Chapter I. Fundamentals on the Notion of Numbers

### Par. 1. Definition of the $\Omega$-rational Numbers as the Limiting Values of Rational Number Sequences.

Let us consider sequences $a_\Omega$ of rational numbers $(a_n)$ $(n = 1,2,3...)$. We say, the sequence $(a_n)$ has the limiting value $a_\Omega$ and we call $a_\Omega$ the $\Omega$-rational number or in short the $\Omega$-number.[5] (Footnote 5 on page 4 of the original: $a_\Omega$ can therefore be taken as a new designation for the sequence $(a_n)$. We want to stress though, that $a_\Omega$ be taken as a number (of a new variety). Our procedings do not differ from the usual approach, according to which certain classes of sequences of rational numbers are being introduced as new numbers.) The sequence $(a_n)$ is called the defining sequence or the component sequence of $a_\Omega$. The various members of the defining sequence are named components of $a_\Omega$. It is appropriate to agree on the following convention: In case the sequence $(a_n)$ is given by an

explicit expression of some kind we often designate (unless misunderstandings are imminent) the $\Omega$-number $a_\Omega$ defined by the sequence by the same explicit expression, by simply replacing the unknown $n$ by the symbol $\Omega$. For instance, the sequence $(n)$ define the $\Omega$-number $\Omega$, the sequence $(n^{-1})$ define the $\Omega$-number $\Omega^{-1}$, the sequence $(n^n)$ define the $\Omega$-number $\Omega^\Omega$, the sequence $(-1)^n$ define the $\Omega$-number $(-1)^\Omega$.
We write:

$$a_\Omega = \operatorname{Lim}_{n\,\Omega} a_n$$

($a_\Omega$ is the limiting value of the sequence $a_n$ for $n = \Omega$.)
We define:

$$a_\Omega = b_\Omega \text{ , if } a_n = b_n \text{ for all natural numbers } n.$$

$$a_\Omega \dotplus b_\Omega = c_\Omega \text{ , if } a_n \dotplus b_n = c_n \text{ for all } n.$$

In similar fashion subtraction, multiplication, and division can be explained in terms of the components. Evidently the division is explained if and only if each component of the denominator is different from zero. The designation for elementary calculative operations fit in with our nomenclature.

The $\Omega$-numbers constitute a commutative ring including the unity-element. The ring contains zero-divisors and thus is not a field: Provided $a_n = 1$ for even $n$, $a_n = 0$ for odd $n$, $b_n = 0$ for even $n$, $b_n = 1$ for odd $n$, then it is true for the thus defined $\Omega$-numbers that $a_\Omega \cdot b_\Omega = 0$, and it is not necessary that either of the two factors be zero. (Footnote 6; The ring of the $\Omega$-numbers contains a subring isomorphous to the field of the rational numbers, if the rational number $a$ is adjoined to the constant sequence $a_n = a$ as a limiting value (for $n = \Omega$).)

Evidently the zero-divisors can be distinguished by a simple property of their component sequences:

An $\Omega$-number is a zero-divisor if and only if the component sequence $(a_n)$ contains at least a single zero.

From this remark it follows that the ring of the $\Omega$-numbers is a quotient-ring: Division by any non-zero-divisor is possible.

An $\Omega$-number is called whole if each of its components is whole and rational. The whole numbers again constitute a ring with the inclusion of zero-divisors. It is easy to observe, that each $\Omega$-rational number is the quotient of two whole $\Omega$-numbers, where the components of the denominator can be taken as natural numbers.

We shall now discuss more extensively a class of $\Omega$-numbers which is of importance in the applications, since most numbers occurring in concrete problems belong into this class. We shall call an $\Omega$-number quasirational, if it can be generated from 1 and $\Omega$ by a finite amount of rational operations ( $\dotplus$, $-$, $:$ ) and "quasirational" operations $\sum$ , $\prod$ . The use of

the symbols $\sum$ and $\prod$ can be explained in the following way:

$$\sum_{\nu=g_\Omega}^{h_\Omega} f_{\Omega,\nu} \quad \text{bzw.} \quad \prod_{\nu=g_\Omega}^{h_\Omega} f_{\Omega,\nu} \quad (g_\Omega, h_\Omega \underset{\text{Whole}}{\text{ganz}}; \; g_n \leq h_n)$$

Above are those $\Omega$-numbers whose component sequences are given by:

$$\sum_{\nu=g_n}^{h_n} f_{n,\nu} \quad \text{bzw.} \quad \prod_{\nu=g_n}^{h_n} f_{n,\nu}.$$

Evidently, the following $\Omega$-numbers are quasirational:

$$\left(1+\frac{1}{\Omega}\right)^\Omega = \prod_{\nu=1}^{\Omega}\left(1+\frac{1}{\Omega}\right); \quad \sum_{\nu=0}^{\Omega}\frac{1}{\nu!}; \quad \sum_{\nu=1}^{\Omega^\Omega}\frac{(-1)^\nu}{\nu}.$$

## Par. 2. Ordering Relations

In anology to the field of the rational numbers (the component field) we also derive a "larger than-" relation for the $\Omega$-numbers. It is appropriate to introduce several ordering relations differing slightly from each other. We start by explaining the positivity or the property corresponding to it. To this end the strongest possible requirement evidently is:

Definition: An $\Omega$-number is named positive, $a_\Omega > 0$, if each of the components $a_n$ is positive.

A weaker requirement would be:

Definition: An $\Omega$-number $a_\Omega$ is called weakly positive, if, finally, all components are positive; we write $a_\Omega \gtrdot 0$.

(The meaning of the words "finally for all components" is: For all components with indices $n > N$ for any natural number $N$ dependent upon the number $a_\Omega$.)

We should remark that analogously to the case of the positivity a "weak" relation is always adjoined to any relation of the $\Omega$-numbers taken over from the domain of the rational numbers. In the "weak" relation the defining requirement has only got to be fulfilled finally (eventually). In the following we shall not always explicitly mention it and accordingly inform the reader now that the weak relation in each case is designated by a dot below the mark characterizing the relation. Let us consider a simple but important example. The weak equality $a_\Omega \doteq b_\Omega$ is valid, if finally (eventually) all components are equal. As is easily demonstrated, this is an equivalence relation. If one designated as zero-numbers all those whose components eventually are equal to zero, then it

is true: Two numbers are weakly equal, if and only if their difference is a zero-number.

From the positivity and the weak positivity we can appropriately explain the relations "larger" and "weakly larger":

Definition: $a_\Omega > b_\Omega$, if $a_\Omega - b_\Omega > 0$. (We may also write $b_\Omega < a_\Omega$).

$a_\Omega \gtrsim b_\Omega$, if $a_\Omega - b_\Omega \gtrsim 0$. (We also write $b_\Omega \lesssim a_\Omega$).

Both relations are transitive and transfer into the usual "larger-" relation for the rational numbers. It is up to the reader to demonstrate the compatability of these two ordering relations with the ring structure. This is also valid for the relation "larger-equal", which can also be introduced by way of the arrangement of the rational numbers:

Definition: $a_\Omega \geqq b_\Omega$ ($a_\Omega$ larger-equal $b_\Omega$), if $a_n$ larger-equal $b_n$ for all components (the corresponding affair in case of $a_\Omega \gtrapprox b_\Omega$, modified by saying, eventually for all components.)

We should mention, that from $a_\Omega \geqq b_\Omega$ it does not have to follow that $a_\Omega$ is either larger or equal to $b_\Omega$, exclusively. The relation must, therefore, be named "larger-equal" and not "larger or equal".

Out of any two rational numbers one is always larger than the other, unless they are equal. For $\Omega$-numbers this total order is not given: the $\Omega$-number $(-1)^\Omega$ is neither equal to zero nor smaller or larger than zero; for this pair of numbers not even one of the weaker relations $\gtrsim$, $\geqq$, $\gtrapprox$, is valid. The absence of total order is, of course, due to the fact that we assigned limiting values to all sequences. Later on we shall also investigate sub-sets of the set of the $\Omega$-numbers, which are totally ordered by means of a suitable relation.

With each $\Omega$-number we can assign a second one, its absolute value $|a_\Omega|$ the components of which are the absolute values $|a_n|$ of the components $a_n$. Evidently it is always true $|a_\Omega| \geqq 0$. If it is true $|a_\Omega| > 0$, we shall say the $\Omega$-number $a_\Omega$ lies apart from zero. Two numbers are called apart from each other (designated by $a_\Omega \# b_\Omega$), if their difference lies apart from zero. According to the criterion for zero-divisors as given in Par. 1 we can say: A number lies apart from zero, if and only if it is different from zero and if it is not a zero divisor.

It presumably is not superfluous to point out that for each pair of $\Omega$-numbers $a_\Omega$ and $b_\Omega$ there is another one $c_\Omega$, which is not apart from either of the first two. $c_\Omega$ is defined by the component sequence $c_{2n-1} = a_{2n-1}$, $c_{2n} = b_{2n}$.

Par. 3. Order of Magnitude (Arrangement According to Magnitude)

The number $\Omega$ has the following property: For each natural number N it is true that $\Omega \geqslant N$. Thus, there is no natural number, which would be (weakly) larger than $\Omega$. For the ring of the $\Omega$-numbers the archimedian axiom in its usual form is not valid; this ring is non-archimedian. We shall presently learn about a suitable replacement for the archimedian axiom. At first let us say that numbers (like $\Omega$) which cannot be reached by the natural numbers shall be called "infinitely large":

Definition: The $\Omega$-number $a_\Omega$ is called infinitely large, if it is true for any natural number N that $a_\Omega \geqslant N$.

Correspondingly:

Definition: An $\Omega$-number $a_\Omega$ is called infinitesimally small, if it is true for any natural number N that $|a_\Omega| \leqslant \frac{1}{N}$.

The proposed replacement for the archimedian postulate is given by: For each two positive $\Omega$-numbers $a_\Omega$, $b_\Omega$ there exists a positive whole $\Omega$-number $N_\Omega$, such that $N_\Omega \, a_\Omega > b_\Omega$.

Proof:
The components $N_n$ are determined such that $N_n \, a_n > b_n$. Here we use the fact that the rational numbers are arranged in the archimedian way.

We shall now turn to the definition of the order of magnitude relations. With the aid of these we shall be in the position to present infinitely large as well as infinitesimally small domains, different from and in relation to each other, in the realm of the infinite and the infinitesimal.

Definition: Given two $\Omega$-numbers $a_\Omega \geqq 0$, $b_\Omega \geqq 0$. Then we call $a_\Omega$ relatively infinitely large as compared with $b_\Omega$, $a_\Omega \gg b_\Omega$, (conversely, we call $b_\Omega$ relatively infinitesimally small as compared with $a_\Omega$, $b_\Omega \ll a_\Omega$), if $a_\Omega \geqq N \cdot b_\Omega$ for any natural number N.

For numbers remote from zero still another relation can be explained.

Definition: Given two not necessarily weakly positive numbers $a_\Omega$, $b_\Omega$ with $a_\Omega \neq 0$, $b_\Omega \neq 0$, then $a_\Omega$ and $b_\Omega$ are called of similar magnitude, $a_\Omega \sim b_\Omega$, if eventually the quotient $a_n/b_n$ lies between certain positive and rational boundaries.

Immediately some conclusions can be drawn, the obvious proofs of which we leave to the reader.

1. The three relations $a_\Omega \gg b_\Omega$, $a_\Omega \sim b_\Omega$, $a_\Omega \ll b_\Omega$

mutually exclude each other. Two $\Omega$-numbers are related by only one of the relations $\gg$ , $\sim$ , $\ll$ . There are examples, in which none of the relations is fulfilled: $a_\Omega = 1$, $b_\Omega = 1 + \Omega(1 + (-1)^\Omega)$.

2. The relations are consistent with the algebraic structure of a ring. We shall mention some easily proved theorems, proved for positive $\Omega$-numbers only for the sake of brevity, although some of the theorems are more generally valid, as the reader will see for himself.

$a_\Omega \sim b_\Omega$ is a relation of equivalence: the classes are named orders of magnitude. From $a_\Omega \sim b_\Omega$ it follows $1/a_\Omega \sim 1/b_\Omega$ . From $a_\Omega \sim b_\Omega$ , $a'_\Omega \sim b'_\Omega$ it follows $a_\Omega + a'_\Omega \sim b_\Omega + b'_\Omega$ and $a_\Omega \cdot a'_\Omega \sim b_\Omega \cdot b'_\Omega$ . For each (positive) $\Omega$-number $a_\Omega$ there are (positive) $\Omega$-numbers $a'_\Omega$ or $a''_\Omega$ , respectively, which are relatively infinitely large or relatively infinitesimally small as compared with $a_\Omega$ . (For a specific case put $a'_\Omega = \Omega \cdot a_\Omega$ and $a''_\Omega = \omega \cdot a_\Omega$ with $\omega = 1/\Omega$ .) A zero number is relatively infinitesimally small as compared with any positive $\Omega$-number. (Therefore zero numbers can be called absolutely infinitesimally small.)

From $a_\Omega \gg a'_\Omega$ , $b_\Omega \gtrsim b'_\Omega$ (or conversely $b_\Omega \sim b'_\Omega$ ) it follows $a_\Omega \cdot b_\Omega \gg a'_\Omega \cdot b'_\Omega$ .

Further theorems can easily be found. It was found unnecessary, however, to go beyond the number of examples of theorems given above, since in any given case one can decide about the magnitude relations by referring back to the definitions. Concerning the nomenclature it should be mentioned that the relation $\gtrsim$ as a "weak" relation actually should be designated by $\gg$ . Since the corresponding strong relation does not play any role, we leave out the dot for simplicity.

The relation $\gg$ gives rise to mentioning another important relation:

Definition: Two $\Omega$-numbers are called of equal magnitude (not to be confused with order of magnitude) if their difference, taken absolutely, is relatively infinitesimally small compared to the absolute value of the former of the numbers:

$$a_\Omega \approx b_\Omega \quad , \text{ if } \left|a_\Omega - b_\Omega\right| \ll \left|a_\Omega\right| .$$

We shall prove at first now that the thus introduced relation is in equivalence relation. The corresponding equivalence classes we shall call "magnitudes". Trivially we have $x_\Omega \approx x_\Omega$ for any $\Omega$-number $x_\Omega$ . As a next step one has to show that from $x_\Omega \approx y_\Omega$ it follows $y_\Omega \approx x_\Omega$ . It is supposed:

$$|x_N - y_N| \leq \frac{1}{n+1} |x_N| \quad \text{für} \quad N \geq N(n),$$

It therefore follows:

$$|x_N - y_N| \leq \frac{1}{n+1}|x_N| \leq \frac{1}{n+1}|y_N| + \frac{1}{n+1}|x_N - y_N|,$$

Therefore:

$$|x_N - y_N| \leq \frac{1}{n}|y_N| \quad \text{für} \quad N \geq N(n),$$

That is $|x_\Omega - y_\Omega| \ll |y_\Omega|$ or, as asserted, $y_\Omega \approx x_\Omega$.

Finally we must prove the transitivity of the relation. Let $x_\Omega \approx y_\Omega$ as well as $y_\Omega \approx z_\Omega$. For $N \geqq N'(n)$ we then have (using the statement proved already):

$$|x_N - y_N| \leq \frac{1}{2n+1}|y_N| \quad \text{und} \quad |y_N - z_N| \leq \frac{1}{2n+1}|y_N|.$$

From this it follows:

$$|x_N - z_N| \leq |x_N - y_N| + |y_N - z_N| \leq \frac{2}{2n+1}|y_N|$$
$$\leq \frac{2}{2n+1}\{|x_N| + |x_N - y_N|\} \leq \frac{2}{2n+1}\left\{|x_N| + \frac{1}{2n}|x_N|\right\}.$$

(The latter inequality comes about similarly to the one derived during the proof of the symmetry of the relation.) We therefore have

$|x_N - z_N| \leqq 1/n\,|x_N|$ or indeed $x_\Omega \approx z_\Omega$.

Observe, that generally from $x_\Omega \approx x'_\Omega$, $y_\Omega \approx y'_\Omega$ it does not follow $x_\Omega + y_\Omega \approx x'_\Omega + y'_\Omega$ as can be seen from the example $x_\Omega = 1$, $x'_\Omega = 1 + \omega$, $y_\Omega = y'_\Omega = -1$. But in case the components of all $\Omega$-numbers occurring are eventually non-negative, that is it is true $x_\Omega \geqq 0$ etc., it follows indeed:

Provided the components of $x_\Omega$, $x'_\Omega$, $y_\Omega$, $y'_\Omega$ are eventually non-negative and provided the following relations are valid

$x_\Omega \approx x'_\Omega$, $y_\Omega \approx y'_\Omega$, then it is true:

$$x_\Omega + y_\Omega \approx x'_\Omega + y'_\Omega.$$

Proof. We show at first: $x_\Omega + y_\Omega \approx x_\Omega + y'_\Omega$:

$$|(x_\Omega + y_\Omega) - (x_\Omega + y'_\Omega)| = |y_\Omega - y'_\Omega| \ll |y_\Omega|,$$

since according to supposition $y_\Omega \approx y'_\Omega$. Eventually it is true: $x_N \geqq 0$, that is $y_N \leqq x_N + y_N$. It is true, too:

$|(x_\Omega + y_\Omega) - (x_\Omega + y'_\Omega)| \ll |x_\Omega + y_\Omega|$, d.h. aber $x_\Omega + y_\Omega \approx x_\Omega + y'_\Omega$.

In much the same manner it can be shown that $x_\Omega + y'_\Omega \approx x'_\Omega + y'_\Omega$ and from the two together there follows the assertion:

Corollary: The corresponding thing is true if all $\Omega$-numbers involved eventually are non-positive numbers.

Whereas the addition of classes of magnitude is only possible under certain added conditions, we have for the multiplication without restriction:

From $x_\Omega \approx x'_\Omega$, $y_\Omega \approx y'_\Omega$ it follows that $x_\Omega y_\Omega \approx x'_\Omega y'_\Omega$.

Proof: Again at first we prove $x_\Omega y_\Omega \approx x_\Omega y'_\Omega$: $|x_\Omega y_\Omega - x_\Omega y'_\Omega| = |x_\Omega| \cdot |y_\Omega - y'_\Omega|$. According to supposition ($y_\Omega \approx y'_\Omega$) from this it follows for $N \geqq N(n)$:

$$|x_N y_N - x_N y'_N| = |x_N| \cdot |y_N - y'_N| \leq \frac{1}{n} |x_N| \cdot |y_N| = \frac{1}{n} |x_N \cdot y_N|,$$

that is $x_\Omega y_\Omega \approx x_\Omega y'_\Omega$. In the same way one proves $x_\Omega y'_\Omega \approx x'_\Omega y'_\Omega$, from which again there follows the above assertion because of the transitivity of the relation $\approx$.

These two remarks permit the representation of the product of two classes of magnitude and under certain conditions also the representation of the sum of two classes of magnitude by the classes of magnitude of the product or the sum, respectively, of representatives.

## Par. 4 Finite Numbers and Real Classes of Numbers

Among the classes of magnitude one plays a distinguished role, namely, the class in which the rational numbers are contained. The $\Omega$-numbers of this class we shall call finite numbers. They do not quite have the role of the real numbers in conventional analysis. In order to arrive at this kind of number we have to demand in addition that the finite number be comparable with the rational numbers. More exactly defined:

Definition of the real $\Omega$-numbers:

An $\Omega$-rational number $r_\Omega$ is called a real $\Omega$-rational number, if the following is true:

a) There are two rational numbers $\varrho_1$ and $\varrho_2$ such that

$$\varrho_1 \leq r_\Omega \leq \varrho_2;$$

b) Let $r'$ be an arbitrary rational number, then one of the following three relations is always valid:

$$r' > r_\Omega \quad oder \quad |r' - r_\Omega| \ll 1 \quad oder \quad r' < r_\Omega.$$

(It may occur therefore that for instance the second relation is only fulfilled or one of the remaining two, as is the case for $r_\Omega = \omega$ and $r' = 0$.)

A class of real numbers (not "$\Omega$-number") or more precisely a Dedekind's class of real numbers is a class of real $\Omega$-rational numbers, whose differences are infinitesimally small.

Evidently the thus defined Dedekind's classes of real numbers are identical with the real numbers defined in the customary way, which can be taken as the classes of defining sequences of $\Omega$-numbers.

It is easy to see, namely, that the "zero-sequences" of common analysis correspond inversely uniquely to the component sequences of the infinitesimally small numbers, since a "zero-sequence has as a limit for $n = \Omega$ an infinitesimally small number and, inversely, an infinitesimally small number has a component sequence, which is a "zero sequence" in the customary sense, as follows directly from the definition of the infinitesimally small numbers.

We should point out, that herewith we do not introduce Dedekind's axiom (or equivalent) and that we therefore do not simply come back to conventional analysis. We rather maintain that we want to calculate with $\Omega$-numbers rather than classes. The introduction of Dedekind's classes of real numbers serves just two purposes. For once this allows comparison of the results of the $\Omega$-analysis with those of the conventional analysis; for another point it is occasionally necessary to have the conception of the class of real numbers on hand, because it is so important from many points of view. Among others one reason is the fact that the classes of real numbers form a completely arranged field.

Often it is appropriate to designate classes of real numbers by suitable representatives, for instance by the class of the infinitesimally small numbers including the zero etc. In order to carry on with nomenclature without introducing possible sources of misunderstanding we shall now introduce the following equivalence relation: $a_\Omega \cong b_\Omega$ (,, a finitely equal to $b_\Omega$"), if their difference is infinitely small. Furthermore it is appropriate to use the designation $\cong$ also for marking that an $\Omega$-number belongs to a class of real numbers. For instance it is true:

$$1 + \omega(-1)^\Omega \cong 1 \qquad \left(\omega = \frac{1}{\Omega}\right)$$

(Finite-equality of two $\Omega$-numbers). Let the class of real numbers, which contains the real $\Omega$-number $(1+\omega)^\Omega$, be called e, then the fact that the number belongs into this class can be expressed as:

$$(1+\omega)^\Omega \cong e.$$

Here we cannot use the equality-sign, since equality is only explained for comparable entities, for instance numbers, whereas the designation $\cong$, on the basis of our agreement, also stands for the fact that a number

belong to a class, as for instance $(1+\omega)^{\Omega} \in e$.

Concerning the nomenclature we further remark, that the "larger-than-relation" between classes of real numbers shall, as is customary, be designated by $>$. This is meaningful, since for each two classes of real numbers a, b, there are elements $a_\Omega \cong a$, $b_\Omega \cong b$, such that either $a_\Omega < b_\Omega$ or $a_\Omega > b_\Omega$. Note that this will not be valid for arbitrary representatives. In general one has to replace the designation $>$ by $\gtrsim$, when going from classes to representatives:

$$e > 5/2 \text{ (for the classes), but}$$

$$(1+\omega)^{\Omega} \gtrsim 5/2 \text{ (for a pair of representatives).}$$

From this remark one can take that generally the $\gtrsim$-relation corresponds to the $>$-relation in conventional analysis. Aside from this relation we also need the stronger $>$-relation in the $\Omega$-analysis. This we shall see in the next section.

## Par. 5. The Conception of the Limiting Value of a Sequence of $\Omega$-Numbers.

If any natural number $p$ is associated with an $\Omega$-number $a_\Omega^{(p)}$, we can talk about a sequence of $\Omega$-numbers. In the same manner double-sequences $a^{(p,q)}$ can be defined.

Definition of the limiting value of a sequence of $\Omega$-numbers.

Let $a_\Omega^{(p)}$ be a sequence of $\Omega$-numbers and let $g_\Omega$ be an infinitely large positive whole number. We then define an $\Omega$-number:

$$\lim_{p=g_\Omega} a_\Omega^{(p)} = a_\Omega^{(g_\Omega)} = b_\Omega$$

by the fact that its defining sequence (component-sequence), is given by:

$$b_n = a_n^{(g_n)} \qquad (n = 1, 2, 3, \ldots).$$

This $\Omega$-number is called the limiting value of the sequence for infinite positive whole $\Omega$-numbers $g_\Omega$.

Note, that $g_\Omega$ has to be strongly positive and that $g_\Omega \gtrsim 0$ does not suffice. This is, because all $g_n$ have to be positive whole numbers in order to form the components of the limiting value. This is the instant, pointed out in the preceding paragraph, at which we need the stronger $>$-relation, whereas in most cases- as seen there - the weaker relation $\gtrsim$ is more important.

A specific application of this definition of a limiting value is possible in the case where the members of the sequence are all rational numbers, in other words do not depend on $\Omega$:

$$a_\Omega^{(p)} = a_p, \; a_p \text{ rational number.}$$

Then

$$\operatorname{Lim}_{p=\Omega} a_{\Omega}^{(p)} = a_{\Omega}$$

just gives the $\Omega$-number whose defining sequence is $a_n$. The definition of the limiting value therefore generalizes the definition of the $\Omega$-rational numbers given in Paragraph 1. It is notable, that each limiting value of a sequence of $\Omega$-rational numbers is again an $\Omega$-rational number.

Each $\Omega$-number has several others associated with it, which are given by the limiting values of the component sequence of $\Omega$ for infinite positive whole $\Omega$-numbers. The relation of the $\Omega$-number $a_{\Omega}$ to those associated with it in the described manner (numbers $a_{g\Omega}$) is elucidated in a special most important case by the following theorem (which at the same time, presents a motivation for the designation "limiting value"):

Principal theorem for classes of real numbers:

If $\operatorname{Lim}_{p=\Omega} a^{(p)} \cong r$ ($r$ is a real number class), then

$$\operatorname{Lim}_{p=g_{\Omega}} a^{(p)} \cong r$$

for each infinite positive whole $\Omega$-number $g_{\Omega}$.

Proof: The supposition that $r \cong a^{(\Omega)}$ be a real number class means by definition: For each rational number $\varrho$ it is clear which of the three relations is valid:

$$r < \varrho, \quad r \cong \varrho, \quad r > \varrho$$

At least one of the relations has to be fulfilled. Let us consider the smaller-relation. Let:

$$a^{(\Omega)} < \varrho.$$

This by definition means: There is $N_o$ such that for $n \geqq N_o$ it is true $a^{(n)} < \varrho$. On the other hand, since $g_{\Omega}$ is infinite there is $N_1$ (which may be dependent on $N_o$) such that $g_n > N_o$ for all $n \geqq N_1$. One therefore has for $n > N_2 = \operatorname{Max}(N_o, N_1)$:

$$a^{(g_n)} < \varrho,$$

that is $a^{(g\Omega)} < \varrho$. The argument is similar for $>$ instead of $<$. Thus $a^{(g\Omega)}$ lies between the same rational boundaries as $a^{(\Omega)}$. For real $a^{(\Omega)}$ this means, that $a^{(g\Omega)}$ is in the same real number-class, q.e.d.

Something else follows from this proof: One may also conclude that from $a_{\Omega} > \varrho$ for a rational number $\varrho$ it follows that $a_{g\Omega} > \varrho$ for all infinite whole positive $g_{\Omega}$; $a_{\Omega}$ is infinite, and thus $a_{g\Omega}$.

This principal theorem of real number-classes, by the way, has a strong connection with the corresponding theorem in conventional analysis. According to the latter, of course, each infinite partial sequence of a Cauchy-convergent sequence also converges and has the same limiting value (in the usual sense).

Let $a_{\Omega}^{(p,q)}$ be a double sequence of $\Omega$-numbers, such that for each given p or q, respectively, there is a sequence of $\Omega$-numbers with the running index p or q, respectively. Furthermore let $h_{\Omega}$ and $g_{\Omega}$ be two not necessarily different infinite positive numbers. Then we can form:

$$\operatorname*{Lim}_{p=g_{\Omega}} a_{\Omega}^{(p,q)} = b_{\Omega}^{(q)} = a_{\Omega}^{(g_{\Omega},q)}$$

for each given q, and following this:

$$\operatorname*{Lim}_{q=h_{\Omega}} b_{\Omega}^{(q)} = b_{\Omega}^{(h_{\Omega})} = a_{\Omega}^{(g_{\Omega},h_{\Omega})}.$$

After this twofold boundary process we obtain that $\Omega$-number $a_{\Omega}^{(g_{\Omega},h_{\Omega})}$, whose components are given by $a_n^{(g_n,h_n)}$. The same $\Omega$-number is evidently obtained if at first one performs the boundary process $q = h_{\Omega}$ and then the process $p = g_{\Omega}$:

The result of two boundary processes is independent of the order in which they are performed. The order in boundary processes is interchangeable.

It seems necessary to say a few words about the difference between the concept of limiting value introduced here and the usual one in conventional analysis. In our structure each sequence of $\Omega$-numbers has a limiting value for each arbitrary infinite positive whole $\Omega$-number. Thereby we also assign limiting values to sequences which in the conventional analysis diverge. The advantage is that in the course of a calculation one does not have to bother about proofs of convergence. One only tests the final result insofar as to see whether it falls into real number-class (if this is at all important). On the other hand the limiting value will always depend upon the boundary in an explicit way. According to the principal theorem above it is assured that all limiting values fall into the same real number-class as soon as the defining rational number-sequence is Cauchy-convergent in the usual sense. It is also assured that this class of limiting values is consistent with the limiting value of the sequence (in the usual sense) in regard to the containment in the rational number-body. This in turn assures the consistency of the results of the new analysis proposed here with those of the conventional analysis, in cases where both are applicable.

This statement seems to be contradicted at first by the fact that the order in which limiting processes are performed is interchangeable, which does not hold in conventional analysis. We shall discuss in this connection the simple example of the twofold sequence of the rational numbers:

$$a^{(p,q)} = \frac{1}{1+\frac{p}{q}},$$

for which, in conventional analysis, it is true that:

$$\lim_{q\to\infty}\lim_{p\to\infty} a^{(p,q)} = 0, \quad \text{aber} \quad \lim_{p\to\infty}\lim_{q\to\infty} a^{(p,q)} = 1.$$

In the case of the presently discussed analysis we have (independent of the order):

$$\operatorname{Lim}_{\substack{p=g_\Omega \\ q=h_\Omega}} a^{(p,q)} = \frac{1}{1+\frac{g_\Omega}{h_\Omega}}.$$

We want to discuss the result of the conventional analysis from our standpoint. If at first we perform the limiting process $p = g_\Omega$ (corresponding to $p \to \infty$) we get:

$$\frac{1}{1+\frac{g_\Omega}{q}}$$

that is for each finite $q = 1, 2, 3, \ldots$ an infinitesimal number. It remains to say that the conventional analysis replaces this infinitesimal number by a different representative, namely 0, such that, naturally, the second limiting process results in zero. In our case it is not permissible to change to a different representative during the calculation, and therefore it is possible that a sequence of infinitesimal numbers (as in our example) leads to a finite limiting value. The result of the conventional analysis can of course always be reproduced. If we want to perform $p \to \infty$, for example, we presuppose $g_\Omega \gg h_\Omega$; one then obtains a limiting value, which belongs into the class, which is given by the limiting value from conventional analysis. Nothing is lost from conventional analysis. In addition one gains the advantage that during a calculation one does not have to think about the interchangeability of limiting processes. In case the final result is a real number, then according to the principal theorum its class is independent of the specific choice of the infinite boundaries insofar as $\Omega$ can be replaced by any other infinite whole positive $\Omega$-number.

In regard to the principal theorem it remains to be remarked that it is important to suppose that the limiting value for $\Omega$ fall into a real number-class, in order to be able to draw the conclusion that all limiting values fall into the same class. In general it is not sufficient to this end to suppose that the limiting value for any infinite whole and positive $\Omega$-number (different from $\Omega$) be real. This is clear from the sequence $a_n = (-1)^n$; Its limiting value is real for the number $2\Omega$, indeed it is equal to 1, but the limiting value for $n = \Omega$ is not real.

Further interesting examples concerning the interchangeability will be given later.

Important examples of sequences are offered by the infinite sums and products introduced earlier. According to the principal theorem it is clear again that each infinite sum (as well as product), which also converges according to conventional analysis, delivers here a real number out of the corresponding class. Note in this connection that also expressions which diverge in conventional analysis make sense here and can be calculated with. Thus we can calculate with "divergent" series, as it was customary up to Euler's times. We shall only quote the simplest and best known example of Bernoulli's: The equation:

$$\sum_{p=1}^{\infty} \frac{1}{p(p+1)} = 1$$

according to Bernoulli can be proven in the following way:

$$\sum_{p=1}^{\Omega} \frac{1}{p(p+1)} = \sum_{p=1}^{\Omega} \left( \frac{1}{p} - \frac{1}{p+1} \right) = \sum_{p=1}^{\Omega} \frac{1}{p} - \sum_{p=2}^{\Omega+1} \frac{1}{p} = 1 - \frac{1}{\Omega+1}.$$

Therefore the asserted equation is valid, independent of the upper limit according to the principal theorem.

We shall now introduce two abbreviations, which are after the nomenclature-convention in Par 1. nearly self evident. Products with all the same factors are written as powers:

$$\prod_{p=1}^{g_\Omega} a_\Omega = a_\Omega^{g_\Omega} \quad (g_\Omega \geq 1).$$

For the products:

$$\prod_{p=1}^{g_\Omega} p \quad (g_\Omega \geq 1)$$

we write in short $g_\Omega !$ (factorial)

After the introduction of the limiting value concept it is now evident why we prefer the strong equality (=) to the weak relation ($\doteq$). As an example we consider the twofold sequence:

$$\delta_{m\,n} = \begin{cases} 1 & \text{für } m = n \\ & \textit{for} \\ 0 & \text{sonst.} \end{cases}$$

It is true: $a_n = \delta_{\Omega\, n} \doteq 0$ für jedes $n$ *for each*

$b_m = \delta_{m\, \Omega} \doteq 0$ für jedes $m$,

But:

$$\lim_{n=\Omega} a_n = \lim_{m=\Omega} b_m = \delta_{\Omega\,\Omega} = 1.$$

If one wanted to perform limiting processes for "weak" equalities, this would result in unreasonable consequences, since from $a_n \doteq 0$ it does not follow for arbitrary $n$ that $a_\Omega \doteq 0$. In the example $a_n = \delta_n$ we even have $a_\Omega = 1$. Such occurrences can be avoided if only the strong equality is used in all considerations concerning the boundary processes. In the example: $a_n = \delta_{\Omega n}$ is not strongly equal to 0. In addition, for strong equality we also have interchangeability of the limiting processes, nonapplicable in weak equalities, as shown by the example.

On the other hand it is usually immaterial to the final result whether strong or weak equality prevails. Therefore one can-if this amounts to a simplification - transfer from the result obtained to the corresponding weakly equal $\Omega$-number. Such a transition is often immaterial in ordering relations ( $\gtrsim$ , $\lesssim$ ), since it only amounts to the addition of a zero-number, that is a number, whose components eventually are zero.

## Par. 6. Remarks Concerning a Possible Extension

We have seen that a whole class of $\Omega$-rational numbers corresponds to each real number in the usual analysis. But there is for instance no $\Omega$-rational number, whose square is equal to two (because, suppose such a number existed, then each of its rational components would have to be such that its square would have to be two, which is impossible.) To be sure, it can be shown that there is for each arbitrary positive $\varepsilon_\Omega$ an $\Omega$-rational number, the square of which, taken absolutely, differs from two by less than $\varepsilon_\Omega$. This certainly is sufficient for all practical purposes and since one is in the position to make a similar statement for all algebraic numbers, one may well be satisfied with this state of affairs particularly when regarding the simplicity of the structure of the number system. Indeed, the present paper is supposed to show that with this approximation one obtains a satisfying construction of an analysis with infinitely small accuracy (meant is probably : infinite accuracy) a construction which avoids some of the faults of the conventional analysis. The following remarks, therefore, are not important for the remainder, but are just meant to round off the matter conceptually.

We shall try to extend the ring of the $\Omega$-numbers in such a way that for instance the extended domain contains the algebraic numbers not only with arbitrary infinitely small accuracy, but exactly. Two approaches are open to this end.

The first line of approach consists of choosing the underlying component-ring in an extended way. (The component-ring for the $\Omega$-rational numbers is the field of the rational numbers). It can be shown - and we want to come back to this point more extensively - that the process, which

led us to the $\Omega$-rational numbers, can generally be applied to partially arranged rings with a distinguished sub-ring of "whole" numbers. If one starts, for instance, from the field of the algebraic or Dedekind-Cantor's real numbers, then one obtains in this manner the rings of the "$\Omega$-algebraic" or "$\Omega$-real" numbers, which contain the ring of the $\Omega$-numbers, but also the field of the real-algebraic numbers as sub-rings. In this connection we like to point out that the process, which led to the $\Omega$-rational numbers, can be iterated.

Another, very interesting possibility of extension arises from the following modification of Cantor's method of fundamental sequences. First we have to define:

Definition: If each positive whole $\Omega$-number $n_\Omega$ is correlated with an $\Omega$-rational number $x_\Omega^{(n_\Omega)}$, such that there exists for each positive $\varepsilon_\Omega$ a whole $\Omega$-number $N_\Omega$ with:

$$|x_\Omega^{(n_\Omega)} - x_\Omega^{(n'_\Omega)}| < \varepsilon_\Omega \quad \text{für} \quad n_\Omega, n'_\Omega > N_\Omega(\varepsilon_\Omega)$$

then this correlation is called a C-sequence:

If there is an $\Omega$-rational number $x_\Omega$, such that:

$$|x_\Omega^{(n_\Omega)} - x_\Omega| < \varepsilon_\Omega$$

For all $n_\Omega > \bar{N}_\Omega$, then one can call this $\Omega$-rational number $x_\Omega$ the limiting value of the C-sequence $x_\Omega^{(n_\Omega)}$ for $n_\Omega \to \infty$. It is easily shown that the C-sequence can at most have one limiting value for $n_\Omega \to \infty$. Conversely, it is also true that each $\Omega$-rational number trivially is the limiting value of a C-sequence. There are C-sequences, for instance:

$$x_\Omega^{(n_\Omega)} = \left(1 + \frac{1}{n_\Omega}\right)^{n_\Omega},$$

which do not have an $\Omega$-rational number as a limit for $n_\Omega \to \infty$. If two C-sequence $x_\Omega^{(n_\Omega)}$ and $y_\Omega^{(n_\Omega)}$ are called equivalent provided for each $\varepsilon_\Omega > 0$ there is a whole $\Omega$-number $N_\Omega$ with:

$$|x_\Omega^{(n_\Omega)} - y_\Omega^{(n_\Omega)}| < \varepsilon_\Omega \quad \text{für} \quad n_\Omega > N_\Omega,$$

then one can extend the ring of the $\Omega$-numbers into the ring of the classes of the equivalent fundamental sequences. In this ring, in which now there is contained an element $\sqrt{2}$ with $\sqrt{2^2} = 2$, each fundamental sequence (C-sequence) converges. This ring also contains a sub-ring isomorphous to the field of the Cantor-Dedekind's real numbers. (In order to acknowledge the latter statement, proceed in the following way: Let, in the usual analysis $r_n \to r$, $r_n$ rational, r real. Then put $r_\Omega^{(n_\Omega)} = \operatorname{Lim}_{n = n_\Omega} r_n$. Then it is true $\lim_{n_\Omega \to \infty} r_\Omega^{(n_\Omega)} = r$.).

The concept of the limiting value for $n_\Omega \to \infty$ shares with the corresponding concept in the usual analysis its disadvantages, for instance the non-interchangeability of the order in limiting processes like this. In the construction of the analysis we therefore shall not make use of this concept of a limiting value. We shall rather restrict ourselves to limiting processes $n = g_\Omega$, thus only using the $\Omega$-rational numbers and not the just obtained upper-ring of the classes of C-sequences. This will be sufficient, because, at that, all quantities are already determined up to arbitrarily small error, which surely meets the requirements of the applications.

Nomenclature

Equality:

$a_\Omega = b_\Omega$, if $a_n = b_n$ for all natural numbers $n = 1,2,3,\ldots\ldots$

Arrangement:

$a_\Omega > b_\Omega$, if $a_n > b_n$ for all $n$

$a_\Omega \geqq b_\Omega$, if $a_n \geqq b_n$ for all $n$.

Lying apart (distant).

$a_\Omega \# b_\Omega$, if $a_n - b_n \neq 0$ for all $n$.

Weak relation:

$a_\Omega R b_\Omega$, if $a_n R b_n$ eventually for all $n$.

($R$ meaning a two-place relation between rational numbers).

Absolute value:

$a_\Omega = |b_\Omega|$, if $a_n = |b_n|$ for all $n$.

Relatively infinitely large (infinitesimally small):

$$a_\Omega \gg b_\Omega \quad (b_\Omega \ll a_\Omega)$$

if $a_\Omega \geqq N \cdot b_\Omega$ for all $N = 1,2,3,\ldots$

Infinitely large (infinitesimally small) numbers:

$$a_\Omega \gg 1 \quad (|a_\Omega| \ll 1).$$

Same order of magnitude:

$a_\Omega \sim b_\Omega$, if $1/N \leqq \dfrac{a_\Omega}{b_\Omega} \leqq N$

for a natural number $N$ and provided $b_\Omega \neq 0$.

Same magnitude:

$a_\Omega \approx b_\Omega$, if $|a_\Omega - b_\Omega| \ll |a_\Omega|$.

Finite-equality (equality up to an infinitesimal error):

$a_\Omega \cong b_\Omega$, if $|a_\Omega - b_\Omega| \ll 1$.

Limiting value for $n = N_\Omega$: $\operatorname{Lim}_{n = N_\Omega} a_\Omega^{(n)} = a_\Omega^{(N_\Omega)}$

with component-sequence $a_n^{(N_n)}$.

## Chapter II Foundations of Analysis

### Par. 1 The Concept of a Function

A function is a mapping:

$$f: D_f \to P^{\Omega}$$

of a sub-set $D_f$ (of the domain of definition) of the ring $P^{\Omega}$ of the $\Omega$-numbers into this ring. All rational functions (with $\Omega$-rational coefficients) are examples of such functions, if the domain of definition consists only of those $\Omega$-rational numbers for which the denominator is not a zero-divisor. Further examples of such functions are:

$$\bar{e}(x) = \sum_{\nu=0}^{\Omega} \frac{x^{\nu}}{\nu!} \quad \text{und} \quad \textit{and} \quad e(x) = \left(1 + \frac{x}{\Omega}\right)^{\Omega}.$$

Other important examples are the functions $f(x) = x^g$ (where $g$ is a whole positive $\Omega$-number).

The most important functions belong into a more special class. In order to define them we first of all need the following definition:

Definition: A set of $\Omega$-rational numbers is called normal, if together with $x_{\Omega}$ also the components $x_n$ belong into the set.

The most important examples of normal sets are the intervals with rational boundaries: The set of the $x_{\Omega}$ with $a \leqq x_{\Omega} \leqq b$ evidently is normal, much the same as the set of the $x_{\Omega}$ with $a < x_{\Omega} < b$ ($a$, $b$, rational in both cases). Also the intervals $0 < x_{\Omega} < \Omega$ and $-\Omega \leqq x_{\Omega} \leqq \Omega$ are normal sets.

Definition. The function $f(x)$ is called normal on the sub-set $M$ of its domain of definition (for short a normal function), if

a) $M$ is a normal set,

b) $f(x_{\Omega}) = \operatorname{Lim}_{n=\Omega} f(x_n)$ for any $x_{\Omega} \in M$.

A normal function is fully determined, if its values for rational numbers from $M$ are given. All examples of functions given in the beginning of the section are normal functions on certain normal sets of their domains of definition. The continuity of a function is explained in the following way:

Definition: A function $f(x)$ is called continuous at the point $x = x_{\Omega}$ of its domain of definition, if for any positive $\Omega$-number $\varepsilon_{\Omega}$ there is a positive $\Omega$-number $\delta_{\Omega}(\varepsilon_{\Omega})$ such that $|f(x_{\Omega}) - f(x'_{\Omega})| < \varepsilon_{\Omega}$ provided only $x'_{\Omega}$ also belongs into the domain of definition and $|x_{\Omega} - x'_{\Omega}| < \delta_{\Omega}(\varepsilon_{\Omega})$. The function $f(x)$ is called uniformly continuous

on a normal sub-set I of the domain of definition, if $\delta_\Omega$ can be chosen in dependent of $x_\Omega$.

Theorem 1. Let f(x) be a normal function on the normal set I. f(x) be continuous in the strict sense at the rational points of I (that is, for any rational $x \in I$ it is true: For any rational, positive $\varepsilon$ there is a rational, positive $\delta(\varepsilon)$ such that $|f(x)-f(x')| < \varepsilon$ for all rational $x' \in I$ with $|x-x'| < \delta$). Then the function is continuous in all points of I.

(Continuation of a continuous function in the strict sense.)

Proof: Let $x_\Omega \in I$ and $\varepsilon_\Omega > 0$, $x_n$ and $\varepsilon_n$ be the associated component sequences such that all $x_n$ belong to I and are rational numbers. According to supposition there are for any n positive rational numbers $\delta_n(\varepsilon_n)$ (which at first may depend upon $x_n$), such that:

$$|f(x_n) - f(x_n')| < \varepsilon_n,$$

if $x_n' \in I$ is rational and $|x_n - x_n'| < \delta_n$. If one defines the $\Omega$-number $\delta_\Omega(\varepsilon_\Omega)$ by the thus obtained components $\delta_n(\varepsilon_n)$, then it is true: $|f(x_\Omega)-f(x'_\Omega)| < \varepsilon_\Omega$, if $|x_\Omega - x'_\Omega| < \delta_\Omega(\varepsilon_\Omega)$. To see this, choose $x_n'$ as the components of the $\Omega$-number $x'_\Omega$.

As a next step we shall prove the theorem of the continuity of the limiting function of a sequence of continuous functions, for which there is no such simple analogous theorem in conventional analysis.

Theorem 2. Let I be a normal set of $\Omega$-numbers and let $f_n(x)$ for any $n = 1, 2, \ldots$ be a function which has rational values for $x \in I$, and which is normal on I and continuous (in the strict sense) (but uniformly continuous in I). If one then defines a function f(x) in I by

$$f(x_\Omega) = \operatorname{Lim}_{n=\Omega} f_n(x_n)$$

then it is true:
f(x) is continuous at any point of I (and uniformly continuous on I).

Proof: We shall give proof for the assertion in brackets. For the remainder of the assertion there is a similar proof. According to supposition it is true for any n:

For any rational $\varepsilon_n > 0$ there is a rational $\delta^{(n)}(\varepsilon_n) > 0$ with:

$$|f_n(x_n) - f_n(x_n')| < \varepsilon_n$$

if the rational numbers

$$x_n, x_n' \in I \quad \text{und} \quad |x_n - x_n'| < \delta^{(n)}(\varepsilon_n).$$

Now choose an arbitrary positive $\Omega$-rational $\varepsilon_\Omega$, the rational components of which are $\varepsilon_n$. Then it is true:

$$|f_n(x_n) - f_n(x_n')| < \varepsilon_n \quad \text{für} \quad |x_n - x_n'| < \delta^{(n)}(\varepsilon_n)$$

Therefore, since this is true for all n:

$$|f(x_\Omega) - f(x'_\Omega)| < \varepsilon_\Omega \quad \text{für/for } |x_\Omega - x'_\Omega| < \delta^{(\Omega)}(\varepsilon_\Omega)$$

where $\delta_\Omega(\varepsilon_\Omega)$ has the components $\delta^{(n)}(\varepsilon_n)$. Since all these components are positive, it is true that $d^{(\Omega)}(\varepsilon_\Omega) > 0$. This is the end of the proof.

The reason why we can carry through this proof is given by the fact that the limiting value (for $n = \Omega$) of a sequence of positive $\Omega$-numbers is again positive (much opposed to conventional analysis.) For an intuitive picture of the situation one could consider the simple example of a sequence of continuous functions, the limiting function of which is not a continuous function in the sense of the conventional analysis, namely the sequence of functions $x^n$ in the interval $0 \leqq x_\Omega \leqq 1$, which constitutes a normal set. The limiting function for $n = \Omega$ is in our case $f(x) = x^\Omega$. This function is equal to zero for $x = 0$, equal to 1 for $x = 1$, but infinitesimally small for $x_\Omega$ with

$$0 < x_\Omega \leq r < 1 \quad (r \text{ rational}).$$

The behaviour of the function is instructive. The function is continuous in an infinitesimal vicinity of one according to the proof above. Let us consider the argument values:

$$x = 1 - \omega^\nu \xi \quad (\xi \text{ positive rational}, \nu \text{ whole} \geqq 0).$$

Then we obtain:

$$x^\Omega = (1 - \omega^\nu \xi)^\Omega = \left(1 - \omega(\xi \omega^{\nu-1})\right)^\Omega,$$

thus $x^\Omega \cong 1$ für/for $\nu > 1$,

but $x^\Omega = e(-\xi) = \left(1 - \frac{\xi}{\Omega}\right)^\Omega$ für/for $\nu = 1$.

For the critical value $\nu = 1$ the function rises like an exponential. Note, that by the definition of continuity $\varepsilon_\Omega$ and $\delta_\Omega(\varepsilon_\Omega)$ are actually of different order of magnitude.

The method of picturing the behaviour of a function in finite regions by "infinite enlargement" of an infinitesimal domain shall often be used in the following.

As we shall see in Par. 4, one cannot without addition prove or even just formulate the analogous theorem for the Cluster-theorem (limiting value) of Bolzano-Weierstrass's for $\Omega$-numbers. It is remarkable on the other hand that one can prove the uniform continuity and the boundedness of a function which is continuous in a finite interval (of finite or infinite extension) without the use of a generalized Cluster-theorem or a corresponding theorem, which we want to show now.

Theorem 3: Let $f(x)$ be defined for all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$

and let it be a continuous, normal function. Then $f(x)$ is uniformly continuous in this interval and bounded by a positive $\Omega$-number $\bar{M}_\Omega$ ( $|f(x)| \leqq \bar{M}_\Omega$ ).

Proof:

a) Proof of the uniform continuity. Provided $f(x_\Omega) = f_\Omega$, then the functions $f_n(x_n)$ are defined by: $f_n(x_n) = f_n$, in the intervals $a_n \leqq x_n \leqq b_n$. Thus we have defined a rationally valued function for all rational numbers in the closed interval $a_n \leqq x_n \leqq b_n$ (footnote on page 22: It follows from the normality of the function $f(x)$, that $f_n$ is uniquely determined by this statement). For this function we want to show now that it is continuous in the customary (strict) sense. Continuity of $f(x)$ means: For any $\varepsilon_\Omega > 0$ there is a $\delta_\Omega(\varepsilon_\Omega) > 0$ (perhaps dependent upon $x_\Omega$), such that:

$$|f(x_\Omega) - f(x'_\Omega)| < \varepsilon_\Omega \quad \text{für} \quad |x_\Omega - x'_\Omega| < \delta_\Omega(\varepsilon_\Omega).$$

If in this expression all quantities are replaced by their rational components, we obtain ( $\delta_\Omega$ ( $\varepsilon_\Omega$ ) having the components $\delta_n$):

$$|f_n(x_n) - f_n(x'_n)| < \varepsilon_n \quad \text{für} \quad |x_n - x'_n| < \delta_n.$$

Since we can suppose $\varepsilon_\Omega$ to be infinitely small, we may choose $\varepsilon_n$ as an arbitrarily small positive rational number. The function $f_n(x_n)$ therefore is continuous (in the strict sense) at any rational point of the interval $a_n \leqq x_n \leqq b_n$, therefore it is uniformly continuous in this same interval: For any positive rational $\varepsilon_n$ we can thus choose $\delta_n = \delta_n(\varepsilon_n)$ independent of $x_n$. If now $\varepsilon_\Omega$ is given as arbitrary positive and if one determines these $\delta_n = \delta_n(\varepsilon_n)$ for the components $\varepsilon_n$ of $\varepsilon_\Omega$, then $\delta_\Omega = \operatorname{Lim}_{n=\Omega} \delta_n$ fulfills uniformly for all $x_\Omega$ of the interval $a_\Omega \leqq x_\Omega \leqq b_\Omega$ the condition of uniform continuity, which was to be shown.

b) Proof of the boundedness.

We can now use the uniform continuity of the function. There is therefore a positive $\Omega$-number $\delta_\Omega$ such that

$$|f(x_\Omega) - f(x'_\Omega)| < 1 \quad \text{für} \quad |x_\Omega - x'_\Omega| < 2\delta_\Omega.$$

Here $\delta_\Omega$ can be chosen such that $N_\Omega = \frac{b_\Omega - a_\Omega}{\delta_\Omega}$ is a whole positive number. Then we have for all positive whole numbers in $0 < M_\Omega \leqq N_\Omega$:

$$f(a_\Omega + M_\Omega \delta_\Omega) = f(a_\Omega) \cdot \sum_{n=0}^{M_\Omega - 1} \{f(a_\Omega + (n+1)\delta_\Omega) - f(a_\Omega + n\delta_\Omega)\}$$

or

$$|f(a_\Omega + M_\Omega \delta_\Omega)| \leq |f(a_\Omega)| + M_\Omega \leq |f(a_\Omega)| + N_\Omega.$$

We shall now show that for any $x_\Omega$ in our interval there is a positive whole $M_\Omega$, such that

$$|x_\Omega - (a_\Omega + M_\Omega \delta_\Omega)| \leq \delta_\Omega.$$

This is because there is a positive rational number $M_n$, such that:

$$|x_n - (a_n + M_n \delta_n)| \leq \delta_n.$$

We have:

$$|f(x_n) - f(a_n + M_n \delta_n)| \leq 1$$

therefore:

$$|f(x_\Omega)| \leq |f(a_\Omega)| + N_\Omega + 1.$$

Herewith we have proven the boundedness of $f(x_\Omega)$ in the closed interval under consideration. (footnote on page 23 of the original: In proof a) we have used the fact that a rationally valued function which is continuous in the interval of the rational numbers (including rational boundaries) in the strict sense, is uniformly continuous in this same interval. This can be proven in the usual way, indirectly. For this one needs the cluster-theorem for rational numbers in the following form: Any bounded infinite set of rational numbers contains a Cauchy-convergent partial sequence. In the proof b) one could use the boundedness of such rationally valued functions, but we have abstained from doing this in order to utilize as little as possible the means of aid from the domain of the rationally valued functions, which appear here only as a special case. end of footnote).

Another important theorem concerning continuous functions in closed intervals is the approximation theorem of Weierstrass's in the conventional analysis, which we like to extend to our numbers and functions.

Theorem 4. (Approximation-theorem). Provided $f(x)$ is continuous and normal in the interval $a_\Omega \leqq x_\Omega \leqq b_\Omega$, then there exists for any $\varepsilon_\Omega > 0$ a function $g(x)$ with:

$$|f(x_\Omega) - g(x_\Omega)| < \varepsilon_\Omega \quad \text{für (for)} \quad a_\Omega \leq x_\Omega \leq b_\Omega$$

and

$$g(x_\Omega) = \operatorname{Lim}_{n=\Omega} g_n(x_n), \quad \text{(wobei (where)} \quad g_n(x_n) \quad \text{in} \quad a_n \leq x_n \leq b_n$$

where $g(x_n)$ are polynomials defined in $a_n \leqq x_n \leqq b_n$ with rational coefficients. $\lceil g(x)$, by the way is a "polynomial of infinitely high degree":

$$g(x) = \sum_{\nu=0}^{N_\Omega} a_\Omega^{(\nu)} x^\nu \rceil .$$

Proof: It is evident that one only has to conduct the proof for the interval $0 \leqq x_\Omega \leqq 1$, since one can pass over from $f(x)$ to $\varphi(\xi) = \varphi\left(\frac{x-a}{b-a}\right) = f(x)$. Then for Bernstein's approximation-polynomials:

$$B_{N_\Omega}(x) = \sum_{k=0}^{N_\Omega} \binom{N_\Omega}{k} x^k (1-x)^{N_\Omega - k} f\left(\frac{k}{N_\Omega}\right)$$

the relation holds:

$$|f(x) - B_{N_\Omega}(x)| \leqq \tilde{\varepsilon}_{N_\Omega} + \frac{M_\Omega}{2\sqrt{N_\Omega}}$$

with

$$\tilde{\varepsilon}_{N_\Omega} \leqq \left| f(x) - f\left(\frac{k}{N_\Omega}\right) \right| \quad \left\{ \text{für} \left| \frac{k}{N_\Omega} - x \right| \leqq \frac{1}{\sqrt{N_\Omega}} \right\} \text{ und } M_\Omega \geqq |f(x)| .$$

(Instead of $\sqrt{N_\Omega}$, $\sqrt[4]{N_\Omega}$ one can use "infinitely close" $\Omega$-numbers, that is numbers the square or fourth power of which deviates infinitely little from $N_\Omega$.) The proof of Bernstein's is of elementary nature and can be performed with our means. (Quite literally it can be taken from, for instance "Lectures about Approximation-theory" by Achieser, Berlin 1953, pages 30-31).

These polynomials already deliver an approximation up to an arbitrarily infinitely small error:

$$|f(x) - B_{N_\Omega}(x)| < \varepsilon_\Omega$$

if only $N_\Omega$ be chosen such that:

$$\tilde{\varepsilon}_{N_\Omega} < \frac{\varepsilon_\Omega}{2}, \qquad \frac{M_\Omega}{2\sqrt{N_\Omega}} < \frac{\varepsilon_\Omega}{2} .$$

If one puts $g(x)$ equal to a suitable Bernstein's approximation polynomial (perhaps one of infinitely high degree):

$$g(x) = B_{N_\Omega}(x) = \sum_{k=0}^{N_\Omega} a_\Omega^{(k)} x^k ,$$

the coefficients of which will generally be $\Omega$-numbers themselves, then it is true:

$$|g(x) - f(x)| < \varepsilon_\Omega$$

(and) und $g(x) = \operatorname{Lim}_{n=\Omega} g_n(x_n)$

with $g_n(x) = \sum_{k=0}^{N_n} a_n^{(k)} x^k$. [$x_n$ belongs into the domain of definition of $g_n$, since $g_n(x_n)$ is the component-sequence of $g_\Omega(x_\Omega)$.] Thus we have shown all there is. (Footnote 8, page 24: Also in case of this theorem we could get at the proof faster by using the Weierstrass's approximation-theorem for rationally - valued functions which are continuous in the stricter sense, which in proof a) let to theorem 3. One would only have to choose the $g_n(x_n)$ as rational polynomials with $|g_n(x_n) - f_n(x_n)| < \varepsilon_n$ for $a_n \leqq x_n \leqq b_n$. In principle, however, it is more interesting to provide direct proof for all of our continuous functions, this is for the same reasons as given in footnote 7 on page 23. End of footnote.)

## Par. 2 Differentiation

It is appropriate to describe the derivative of a function $f(x)$ of the argument $x$ as the difference-quotient:

$$f'_{\bar\omega}(x) = \frac{1}{\bar\omega}\{f(x+\bar\omega) - f(x)\}$$

with sufficiently small increments $\bar\omega$ with $|\bar\omega| > 0$. Such a difference-quotient will occasionally be called a differential-quotient associated with the differential $dx = \bar\omega$. For the applications these differential-quotients are generally sufficient, although in general they are subject to alterations if one changes the differential. Therefore it is reasonable also to introduce the concept of a derivative, which - geometrically speaking- describes the slope of the tangent, whereas the differential-quotient describes the slope of the secant which cuts through adjacent points of the curve. Whereas the differential-quotients always exist, if only $x$ and $x + \bar\omega$ belong to the domain of definition, one cannot always state this for the derivative. We define:

Definition: The function $f(x)$ is called differentiable for the argument $x$, if for any positive $\Omega$-number $\varepsilon_\Omega$ there is a positive $\Omega$-number $\delta_\Omega$, such that:

$$|f'_{\bar\omega}(x) - f'_{\tilde\omega}(x)| < \varepsilon_\Omega \quad \text{(if only) wenn } x,\ x+\bar\omega,\ x+\tilde\omega$$

lie in the domain of definition and if $0 < |\bar\omega| < \delta_\Omega$, $0 < |\tilde\omega| < \delta_\Omega$.

If there exists an $\Omega$-rational number $f'(x)$, such that for any $\varepsilon_\Omega > 0$ there is a $\delta_\Omega > 0$ with $|f'(x) - f'_{\bar\omega}(x)| < \varepsilon_\Omega$ for $|\bar\omega| < \delta_\Omega$, then we call $f'(x)$ the derivative of $f(x)$ at $x$.

The derivative is unique, if it exists. From the existence of the derivative follows the differentiability.

For the derivatives the known rules are valid, which can be proven as usual (Chain-rule, product-rule, etc.) Here, in addition, these rules are also valid for infinitely large as well as infinitesimally small values of the derivatives.

It is important also to consider these rules in connection with the differential-quotients. As an example we quote the chain-rule. We suppose that all arguments lie in their respective domains of definition:

Let $h(x) = f(g(x))$. Then one calculates at once:

$$h'_{\bar{\omega}}(x) = f'_{\tilde{\omega}}(g(x)) \cdot g'_{\bar{\omega}}(x),$$

where

$$\tilde{\omega} = g'_{\bar{\omega}}(x)\,\bar{\omega}.$$

We now quote the differential-quotients for some simple functions:

$$f(x) = c_\Omega, \quad f'_{\bar{\omega}}(x) = 0, \qquad f'(x) = 0,$$
$$f(x) = x, \quad f'_{\bar{\omega}}(x) = 1, \qquad f'(x) = 1,$$
$$f(x) = x^2, \quad f'_{\bar{\omega}}(x) = 2x + \bar{\omega}, \qquad f'(x) = 2x.$$

One can show that any rational function is differentiable in all places of its domain of definition (that is with the exception of poles) and has known derivatives. In case of a rational function with rational coefficients the values of the differential-coefficient deviate infinitely little from the values of the derivative and the derivative is a rational number if $x$ is rational.

As usual one can also show, that the continuity follows from the differentiability.

Similar to the case of the continuity (theorem 2) the differentiability also gets transferred to the limiting function of a sequence of functions:

Theorem 5: (The differentiability of the limiting function of a sequence of differentiable functions)

Let $f(x_\Omega) = \operatorname{Lim}_{n=\Omega} {}^{(n)}f(x_n)$ and ${}^{(n)}f(x_n)$ be rationally valued at $x_n$ and differentiable in the usual sense [let it have the derivative ${}^{(n)}f'(x_n)$.] Then $f(x_\Omega)$ is differentiable at $x_\Omega$ [and has the derivative:

$$f'(x_\Omega) = \operatorname{Lim}_{n=\Omega} {}^{(n)}f'(x_n)].$$

Proof: According to supposition it is true for any $n$: given rational $\varepsilon_n > 0$ there is a $\delta_n(\varepsilon_n) > 0$, such that:

(*) $$|{}^{(n)}f'_{\bar{\omega}^{(n)}}(x_n) - {}^{(n)}f'_{\tilde{\omega}^{(n)}}(x_n)| < \varepsilon_n$$

for $0 < |\bar{\omega}^{(n)}|, |\tilde{\omega}^{(n)}| < \delta_n$ ($\bar{\omega}^{(n)}, \omega^{(n)}$ rational). Let an arbitrary positive $\varepsilon_\Omega$ be given; it has to be shown that there is a positive $\delta_\Omega(\varepsilon_\Omega)$ such that for all $\bar{\omega}_\Omega, \tilde{\omega}_\Omega$ with $0 < |\bar{\omega}_\Omega|, |\tilde{\omega}_\Omega| < \delta_\Omega$ it is true that

(**) $$|f'_{\bar{\omega}_\Omega}(x_\Omega) - f'_{\tilde{\omega}_\Omega}(x_\Omega)| < \varepsilon_\Omega .$$

Let the following expression be defined:

$$\delta_\Omega(\varepsilon_\Omega) = \operatorname*{Lim}_{n=\Omega} \delta_n(\varepsilon_n) .$$

For the components on both sides of (**) the inequalities (*) are valid; thus we have proven the asserted differentiability.

If the derivatives of all approximation-functions are existent, one can repeat the proof with ${}^{(n)}f'(x_n)$ instead of ${}^{(n)}f'_{\omega_n}(x_n)$, in order to gain the limiting relation for the derivative $f'(x_\Omega)$.

This theorem in conjunction with the approximation theorem 4 give another theorem: Any continuous, normal function can be replaced with arbitrarily good accuracy by an infinitely often differentiable function.

Note that a differentiable function does not have to have a derivative at each point (the "derivative", namely, could be"irrational"). Most of the time relatively infinitely small errors are not important, wherefore one often can explain the derivative by the class of magnitude of the differential-quotients are sufficiently small differentials. Rules for the calculation with classes of magnitudes of this kind have been presented earlier (Chapter I, Par. 3).

## Par. 3 Integration

As in Riemann's work the intuitive background for the description of the integral is given by the limiting value of the areas associated with approximating stair-case polygons. Let two $\Omega$ -numbers $a_\Omega, b_\Omega$ with $a_\Omega < b_\Omega$ be given as the limits of the interval. Then a sequence of positive $\Omega$ -numbers $\omega^{(n)}_\Omega \leqq \omega'$ is called the subdivision T of degree of fineness $\leqq \omega'$. For the $\Omega$ -numbers it is true:

$$\sum_{n=1}^{\varepsilon_\Omega} \omega^{(n)}_\Omega = b_\Omega - a_\Omega .$$

The numbers:

$$a_\Omega + \sum_{n=1}^{N_\Omega} \omega_\Omega^{(n)} \qquad (0 < N_\Omega \leq g_\Omega)$$

are called division-points. The "number $g_\Omega$ of division-intervals" is of course dependent on the subdivision.

Let $f(x)$ be a function, the domain of definition of which includes all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$. Then the expression:

$$(T)\int_a^b f(x)\,dx \underset{\text{def}}{=} \sum_{n=1}^{g_\Omega} f\Big(a_\Omega + \sum_{\nu=1}^{n} \omega_\Omega^{(\nu)}\Big)\omega_\Omega^{(n)}$$

is called the subdivision-sum of the function $f(x)$ in the interval associated with the subdivision $(T)$.

Definition: The function $f(x)$ is called integrable in the interval $a_\Omega \leqq x_\Omega \leqq b_\Omega$ if all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$ belong to its domain of definition and if for any positive $\varepsilon_\Omega$ there is a positive $\delta_\Omega$, such that:

$$\left|(T_1)\int_a^b f(x)\,dx - (T_2)\int_a^b f(x)\,dx\right| < \varepsilon_\Omega$$

for any two subdivisions $T_1$, $T_2$, the degree of fineness of which is at most $\delta_\Omega$. If there is an $\Omega$-number $\mathfrak{J}$, such that for any $\varepsilon_\Omega > 0$ there is a $\delta_\Omega > 0$ with:

$$\left|\mathfrak{J} - (T)\int_a^b f(x)\,dx\right| < \varepsilon_\Omega$$

for all subdivisions $T$, the degree of fineness of which is at most $\delta_\Omega$, then we call $\mathfrak{J}$ the integral of the function $f(x)$ in the interval $a_\Omega \leqq x_\Omega \leqq b_\Omega$ and write:

$$\mathfrak{J} = \int_a^b f(x)\,dx.$$

From the existence of the integral follows the integrability. Since the integrals can be replaced by subdivision-sums associated with sufficiently fine steps to arbitrarily good approximation, we can get along with the subdivision-sums in the applications and most of the time we do not have to consider integrals.

(Footnote on page 27 of the original: This means for the components: Given is a twofold sequence $\omega_\nu^{(m)}$ of positive rational numbers (m, n natural numbers) and a sequence $g_n$ of natural numbers, such that $\omega_\nu^{(m\nu)} \leqq \omega_\nu$ and $\sum_{n=1}^{g_\nu} \omega_\nu^{(n)} = b_\nu - a_\nu$, for all natural numbers $\nu$.

End of footnote).

The statement made in the derivation is also valid for integrals. (statement about the replacement of integrals or the subdivision-sums by their classes of magnitude).

An important point of view for the applications is the fact that the limits of integration (also their absolute value) may be infinite, which occasionally permits the treatment of improper integrals without additional considerations.

In the definition of the subdivision-sums one can, as usual, also use the values of the function in the interior of a partial interval, instead of choosing the value of the function at the boundary value on the right hand side. In the interest of brevity we shall abstain from discussing this extension of the definition which can be performed without further difficulty.

The most important type of a subdivision is formed by the uniform subdivisions:

$$\omega = \overset{\sigma}{(u)}\omega \qquad \text{(for all)} \quad n$$

the degree of fineness of which can be chosen arbitrarily small. In this case we shall write ($\omega'$) instead of (T).

The principal theorems concerning integrals can be taken (mutatis mutandis) from Riemann's theory as far as formulation as well as proof goes. The transfer of the proofs we can leave to the reader in most cases. The subdivision-sums as well as the integrals are linear functionals of the integrand-function. Just as directly it can be seen, that:

$$(T_1)\int_a^b f(x)\,dx + (T_2)\int_b^c f(x)\,dx = (T_3)\int_a^c f(x)\,dx$$

Where $T_3$ is the subdivision of the total interval which came about by the composition of the subdivisions $T_1$ and $T_2$:

$$\sum_{n=1}^{g_\Omega+h_\Omega} f\Big(a_\Omega + \sum_{\nu=1}^{n}\omega_\Omega^{(\nu)}\Big)\omega_\Omega^{(n)} = \sum_{n=1}^{g_\Omega} f\Big(a_\Omega + \sum_{\nu=1}^{n}\omega_\Omega^{(\nu)}\Big)\omega_\Omega^{(n)} + \sum_{n=1}^{h_\Omega} f\Big(b_\Omega + \sum_{\nu=1}^{n}\widetilde{\omega}_\Omega^{(\nu)}\Big)\widetilde{\omega}_\Omega^{(n)}$$

with

$$\widetilde{\omega}_\Omega^{(n)} = \omega_\Omega^{(g_\Omega+n)} = \operatorname*{Lim}_{m=\Omega}\omega_m^{(g_m+n)}.$$

For integrals the corresponding theorem holds true.

The fundamental theorem of the differential- and integral calculus holds true. For brevity we shall only quote it here for the case of a uniform subdivision:

$$F'_{\omega'}(x) = f(x) \quad \text{für} \quad F(x) = (\omega')\int_a^x f(x)\,dx.$$

For

The above is a statement concerning differential-quotients for the differential $dx = \omega'$ and concerning the subdivision-sums. Correspondingly the fundamental theorem is also valid for derivatives and integrals.

The continuous functions are the most important examples of integrable functions.

Theorem: Let $f(x)$ be normal and continuous for all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$. Then $f(x)$ is integrable in this interval.

As usual, the proof is based on the uniform continuity (theorem 3).

## Par. 4. Intermediate Value-and Mean-Value-Theorems

In general one cannot expect that a function, which is continuous for all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$ with $f(a_\Omega) < 0$ and $f(b_\Omega) > 0$, takes on the value 0 at least at one $\Omega$-rational intermediate point. This can be shown with the example $f(x) = x^2-2$, $a = 0$, $b = 2$. But it is evident, that for any positive $\Omega$-rational $\varepsilon_\Omega$ one can find an $x_\Omega$ in the interval such that $|f(x_\Omega)| < \varepsilon_\Omega$. This we want to prove now in general for continuous functions.

Ahead of time we like to remark, that here we cannot prove the intermediate value-theorem with the aid of the cluster-theorem of Weierstrass's-Bolzano's - as in conventional analysis. The latter theorem, namely, is not valid here without modification, as can be seen already from the sequence $a_n = 1/n$, which does not have any cluster in the sense that almost all numbers of the sequence lie in arbitrary (also infinitesimally small) vicinity of this point. Cluster-theorem only becomes provable if one makes statements about a generalized sequence, which associates an $\Omega$-rational number $a_{n\Omega}$ with any positive whole $\Omega$-number $n_\Omega$. This, however, we do not want to touch upon.

Here we shall prove the intermediate value-theorem by way of the approximation-theorem (theorem 4, Par. 2).

Theorem 6: (Intermediate value theorem). Let $f(x)$ be normal and continuous for all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$ and let:

$$f(a_\Omega) < 0 < f(b_\Omega).$$

Then for any $\varepsilon_\Omega > 0$ there is an $x_\Omega$ out of the interval, such that $|f(x_\Omega)| < \varepsilon_\Omega$.

Proof: It is sufficient to consider such $\varepsilon_\Omega$ for which

$$f(a_\Omega) < -\varepsilon_\Omega < 0 < \varepsilon_\Omega < f(b_\Omega).$$

We now choose according to theorem 4 an approximation-polynomial $g(x)$, such that:

$$|f(x_\Omega) - g(x_\Omega)| < \tfrac{1}{2}\varepsilon_\Omega$$

for all $x_\Omega$ out of the interval. Furthermore let $g_n(x_n)$ be the rational polynomials as determined in theorem 4 with:

$$\operatorname*{Lim}_{n=\Omega} g_n(x_n) = g(x_\Omega).$$

According to supposition we have:

$$g(a_\Omega) < -\tfrac{1}{2}\varepsilon_\Omega < 0 < \tfrac{1}{2}\varepsilon_\Omega < g(b_\Omega),$$

therefore

$$g_n(a_n) < -\tfrac{1}{2}\varepsilon_n < 0 < \tfrac{1}{2}\varepsilon_n < g_n(b_n).$$

Therefore according to the conventional intermediate value-theorem there exists for rational polynomials a rational number $r_n$ with

$a_n < r_n < b_n$, such that: $\underline{|g_n(r_n)| < \tfrac{1}{2}\varepsilon_n.}$

Thus for $r_\Omega = \operatorname{Lim}_{n=\Omega} r_n$ it is true $a_\Omega < r_\Omega < b_\Omega$ and $|g(r_\Omega)| < \tfrac{1}{2}\varepsilon_\Omega$,

and therefore according to the triangle-inequality :

$$\underline{|f(r_\Omega)| < \varepsilon_\Omega}$$

which had to be shown.

Under the same suppositions as in case of the intermediate value-theorem we can also prove the following theorem 7:
(Theorem of the maximum and the minimum of a continuous function). Let f(x) be normal and continuous for all $x_\Omega$ with $a_\Omega \leqq x_\Omega \leqq b_\Omega$. Then for any positive $\varepsilon_\Omega$ there are $\Omega$-numbers $c_\Omega, d_\Omega$ out of the interval with:

$$f(c_\Omega) - \varepsilon_\Omega \leq f(x_\Omega) \leq f(d_\Omega) + \varepsilon_\Omega$$

for all $x_\Omega$ out of the interval.

These theorems are mainly applied in the proof of the mean-value-theorems, which we shall turn to now.

Theorem 8: (Mean-value-theorem of integral calculus). Let f(x), g(x) be normal and continuous in $a_\Omega \leqq x_\Omega \leqq b_\Omega$, and let $g(x) \geqq 0$. Then (with a certain subdivision) there is for any $\varepsilon_\Omega > 0$ a point $y_\Omega$ in the interval, such that:

$$\left|{}^{(T)}\int_a^b f(x)\,g(x)\,dx - f(y_\Omega)\cdot {}^{(T)}\int_a^b g(x)\,dx\right| < \varepsilon_\Omega.$$

Using the theorem of the maximum and the intermediate value-theorem the proof goes as usual (see for instance Courant, Differential- und Integralrechnung I, p. 102).

This theorem formulated here for subdivision-sums is also valid for integrals in the corresponding modification. As a consequence one can prove the mean-value-theorem of differential calculus, provided the derivative or the differential-quotient exist for a given differential in the interval.

*

## Chapter III Some Examples and Applications

### Par. 1 The Elementary Functions

As first examples for the modifications and extension of the analysis proposed in this paper we shall treat the so-called elementary functions, that is the exponential function, the logarithm, the trigonometric functions, and the hyperbolic functions. It will become apparent, that these functions can even be obtained through a special class, namely that of the "quasirational" functions. Under quasirational functions is understood (similar to the quasirational numbers defined in Chapter I, Par. 1) a function which can be generated in an explicit expression from 1, and x alone by rational and quasirational operations ($\sum$, $\prod$). The quasirational functions are all continuous and normal functions. (Application of theorem 2).

At first we shall define a number of functions:

(1) $$\underset{(N_\Omega)}{e}(x) = \left(1 + \frac{x}{N_\Omega}\right)^{N_\Omega},$$

(2) $$\underset{(N_\Omega)}{\overline{e}}(x) = \sum_{n=0}^{N_\Omega} \frac{x^n}{n!},$$

(3) $$\underset{(N_\Omega)}{\sin} x = \sum_{n=0}^{N_\Omega} \frac{x^{2n+1}}{(2n+1)!}(-1)^n,$$

(4) $$\underset{(N_\Omega)}{\cos} x = \sum_{n=0}^{N_\Omega} \frac{x^{2n}}{(2n)!}(-1)^n,$$

(5) $$\underset{(N_\Omega)}{\operatorname{arc\,tg}} x = \sum_{n=0}^{N_\Omega} \frac{(-1)^n x^{2n+1}}{2n+1},$$

(6) $$\begin{cases} \underset{(N_\Omega)}{\log} x = \sum_{n=\bar{\Omega}}^{N_\Omega+\bar{\Omega}} \frac{1}{n+\bar{\Omega}} = (\bar{\omega}) \int_1^x \frac{dx}{x}, \\ \left(\bar{\Omega} = \frac{N_\Omega}{x-1}, \text{ ganz; whole } \bar{\omega} = \frac{1}{\bar{\Omega}}\right). \end{cases}$$

Here $N_\Omega$ may be an arbitrary infinitely large positive and whole $\Omega$-number. The functions 1 through 5 are defined for any $N_\Omega$ and for all $\Omega$-rational $x_\Omega$ (even for all $x_\Omega = y_\Omega + i \cdot z_\Omega$, $i^2 = -1$, $y_\Omega$, $z_\Omega$, $\Omega$-rational), since no divisions by x occur. All six types of functions are quasirational. One can easily verify simple relations like:

$$\underset{(2N_\Omega+1)}{\bar{e}}(i\cdot x)=\underset{(N_\Omega)}{\cos}\,x+i\cdot\underset{(N_\Omega)}{\sin}\,x.$$

Evidently the quasirational functions defined above do not have (for all $x_\Omega$) the functional properties of the exponential function, of the trigonometric functions, etc. From the point of view of the conventional analysis one can only expect the validity of such relations in those parts of the domain of definition, for which they can be proven in conventional analysis, too, namely for $x_\Omega \sim 1$. Before we go on to show, that here we can prove even more, we like to remark, that the proofs found in any textbook of analysis without trouble furnish the following relations:

$$\underset{(N_\Omega)}{\bar{e}}(x)\cong\underset{(N_\Omega)}{e}(x);\qquad \underset{(N_\Omega)}{e}(\log x)\cong x;\qquad \underset{(N_\Omega)}{e}(x+y)\cong\underset{(N_\Omega)}{e}(x)\underset{(N_\Omega)}{e}(y)$$

for all real $\Omega$-rational $x_\Omega, y_\Omega$ (that is for all $\Omega$-numbers, which are finite and comparable to the rational numbers.) In the same way follow the addition-theorems for the trigonometric functions and for the functional equation of the logarithm.

In all these cases two facts add to the construction of our analysis: For once the infinitesimal error occurring in all these relations can be estimated and given exactly. Secondly, these relations often are valid for certain infinite arguments, too. We shall explain this in the case of the exponential function and its derivative (For differential-quotients at infinitesimal differentials one could carry through similar considerations):

$$\underset{(N_\Omega)}{e'}(x)=\underset{(N_\Omega)}{e}(x)\left(1+\frac{x}{N_\Omega}\right)^{-1}$$

and

$$\underset{(N_\Omega)}{\bar{e}'}(x)=\underset{(N_\Omega)}{e}(x)-\frac{x^{N_\Omega}}{N_\Omega!}.$$

It is evident: the function and the derivative coincide within an exactly giveable, infinitesimal error. For arbitrarily given $x_\Omega$ one can always determine an $N_\Omega$ such that $e$ and $e'$ (or $\bar{e}$ and $\bar{e}'$) are of equal magnitude, that is such that the difference between them is infinitely small. If we talk about classes of magnitude, as they have been investigated in Chapter I, Par. 3, one can exactly obtain the functional property for functions, the values of which are classes. One knows therefore, that for a given problem only the classes of magnitude are important and not the exact values of the $\Omega$-numbers, such that one can transfer to this other kind of function.

Now we can also introduce a generalized power by the composition of the exponential function and the logarithm also by means of quasirational functions up to arbitrary infinitely small error:

$$\underset{(N_\Omega)}{a^b}=\underset{(N_\Omega)}{e}\left(b\cdot\underset{(N_\Omega)}{\log}\,a\right).$$

## Par. 2. $\delta$ -Functions

The best known example, for which it is appropriate to use infinitesimal as well as infinite numbers in the description, is Dirac's $\delta$-function, which in one point is supposed to be infinitely large, in all other points is supposed to vanish, and which has a finite integral. Such a function cannot be explained on the basis of classical concepts, and a strict argument has, as is well known, been put forward by L. Schwartz in the framework of an extended concept of function as given in his distribution-theory. It appears, however, that one meets the importance of this function for the applications in a better way, if one introduces it with the help of the modifications of the analysis as proposed in this paper in the following way:

Definition: An integrable function $\delta(x)$ is called of the kind $\delta$, if there is a positive, infinitesimal $\Omega$-rational number $r_\Omega$, such that: (I) $\delta(x)$ is infinitesimal for $|x_\Omega| > r_\Omega$, such that:

$$(T)\int_a^b \delta(x)\,dx$$

is infinitesimal for $r_\Omega < a < b$ or $r_\Omega < -b < -a$.

(II) $(T)\int_{-a}^{+a} \delta(x)\,dx \approx i$ for $a \geqq r_\Omega$ and sufficiently fine subdivisions (T). We then say, the function of the kind $\delta$, namely $\delta(x)$, has a breadth $\leqq r_\Omega$. An example of a function of kind $\delta$ is:

$$\delta(x) = \frac{1}{\pi}\,\frac{\Omega}{1+\Omega^2 x^2},$$

Where $r_\Omega^2$, for instance, differs relatively infinitely little from $\omega$. There are but many significantly different functions of the kind $\delta$. In praxi this is an advantage, because the $\delta$-function can be suited to the specific problem.

Before quoting more examples of functions of the kind $\delta$, we want to prove that all functions of the kind $\delta$ have the following important property:

Theorem 9: Let $\delta(x)$ be a non-negative function of the kind $\delta$ with a breadth $\leqq r_\Omega \ll 1$ and let $f(x)$ be defined in $-a \leqq x \leqq +a$ in the sense of the usual analysis ($|a| > \varrho$ for positive rational $\varrho$), let $f(x)$ be continuous in the usual sense and let it be bounded by a rational M. (In case $f(x)$ is only defined for all rational numbers of the interval, extend the function by continuous continuation - Chapter II, Par 1 - to all $\Omega$-rational numbers of the interval). Then it is true:

$$(T)\int_{-a}^{+a} \delta(x)\,f(x)\,dx \cong f(0),$$

provided the subdivision is sufficiently fine.

Proof: Because of the boundedness of f(x) the two integrals:

$$(T)\int_{-a}^{-r} \delta(x)\, f(x)\, dx, \qquad (T)\int_{r}^{a} \delta(x)\, f(x)\, dx \qquad (a \geq r)$$

are infinitesimally small. (The subdivision T has to be chosen such that -r +r are division points). The remaining integral:

$$(T)\int_{-r}^{+r} f(x)\, \delta(x)\, dx$$

is left to consideration. In that interval it is true:

$$f(x) \cong f(0).$$

because the function is continuous in the usual sense and because the interval is infinitesimally small.
The mean-value-theorem of integral calculus (theorem 8) gives:

$$(T)\int_{-r}^{+r} f(x)\, \delta(x)\, dx \cong f(0), \quad \text{weil / because} \int_{-r}^{+r} \delta(x)\, dx \cong 1,$$

and from this the assertion follows.

It is easy to see how the suppositions can be lessened in specific cases: f(x) does not necessarily have to be continuous in the usual sense. It is sufficient, if f(x) has infinitely small deviations in the interval $-r \leqq x \leqq +r$. Also the demand for boundedness can be lessened. Care should just be taken, to have the integrals infinitely small in the exterior intervals.

The folding-property can be violated, if, for instance, f(x) itself is function of the kind $\delta$ with the same width as $\delta$ (x). Indeed, it is true for the function (1) of the kind $\delta$, as is easily verified:

$$\int_{-a}^{+a} \delta^2(x)\, dx \approx \tfrac{1}{2}\, \delta(0).$$

The following principle of generation holds valid for functions of the kind $\delta$: Let g(x) be defined in the usual sense (that is let g(x) be rationally valued at rational points, let it be non-negative, and let its integral from $-\infty$ to $+\infty$ (in the usual sense) have the value 1, and let it have its maximum at the origin and let it decline monotonously from the origin to both sides.) Then the following functions (for any infinitely large, whole, and positive $N_\Omega$) are of the kind $\delta$:

$$\delta(x_\Omega) = N_\Omega \cdot g(x_\Omega N_\Omega).$$

/Here the functional value is, of course, determined as the Lim for $n = \Omega$ of the sequence $N_n \cdot g(x_n \cdot N_n)$/.

By this principle we obtain from:

(2) $$g(x) = \frac{1}{\pi}\,\frac{1}{1+x^2}$$

for $n = \Omega$ the function (1) of the kind $\delta$ and for $n = N_\Omega$ the functions:

(3) $$\delta(x) = \frac{1}{\pi}\,\frac{N_\Omega}{1+x^2 N_\Omega^2}.$$

The bell-shaped function g(x) may contain $\Omega$ explicitely. This is true, too, for the following "box-shaped" function:

(4) $$g(x) = \frac{c}{1+(1+x^2-a^2)^\Omega}$$

with $c = 1/2\ a$, a rational, $|a| < 1$.

Already from the definition it follows that the indefinite integral of a function of the kind $\delta$:

$$\eta(x) = \int_0^x \delta(y)\,dy$$

is a "jump-function", which of course, performs in a continuous way a jump of a height which is infinitesimally different from 1 in an infinitesimal vicinity of the number 0. If the function $\delta(x)$ is differentiable, the derivative is a "dipol-function". For the $\delta$-function (1) one obtains for instance:

$$\eta(x) \cong \frac{1}{\pi}\,\underset{(\Omega)}{\operatorname{arc\,tg}}\,\Omega x$$

and

$$\delta'(x) = -\frac{2}{\pi}\,\frac{\Omega^2 x}{(1+\Omega^2 x^2)^2}.$$

This dipole-function $\delta'$ has its extreme for $x = \pm \frac{1}{\Omega\sqrt{3}}$.

The examples of $\delta$-functions considered here are symmetrical: $\delta(-x) = \delta(x)$. This class of functions of the kind $\delta$ is most important because its derivative-functions correspond to antisymmetric dipoles.

One further class important for the applications is that of the orthogonal normalized functions.

## Par. 3 Power Series, Particularly Their Behavior at the Limit of Convergence

In some instances we encountered already power series (or polynomials of infinite degree). We now want to take a systematic look at them. At first we remark that an expression:

The coefficients $a_\Omega^{(n)}$ of which may be $\Omega$-rational numbers. The expression itself is an $\Omega$-rational number for all $\Omega$-rational numbers $x_\Omega$, wherefore it always "converges". Other questions are the ones concerning the change in the expression after alteration of the upper limit

of summation, or the one concerning the exact (up to infinitesimally small or relatively infinitesimally small errors) representation of functions by power series like this.

As a simple and characteristic example we consider the geometrical series:

$$\sum_{n=0}^{N_\Omega} x^n.$$

For all $x = x_\Omega$ apart from 1 (that is those, the components of which are all different from 1) one can represent this series exactly by:

$$\sum_{n=0}^{N_\Omega} x^n = \frac{1 - x^{N_\Omega+1}}{1 - x}.$$

From this we can take: For the $x_\Omega$ the geometrical series is equal (up to an infinitesimal error) to the function:

$$(1 - x)^{-1}, \quad \text{for which} \quad |x^{N_\Omega+1}| \ll |1 - x|.$$

This is true for all complex x, for which there exists a rational number r with $0 < r < 1$, such that $|x_\Omega| \leqq r$. On the other hand, for x with $x_\Omega \geqq r > 1$ one has, that the value of the geometrical series is infinitely large, such that the series does not represent the function there, as was not to be expected otherwise. We mention this fact explicitly here in reference to the often heard objection against calculating with divergent expressions (compare in this connection Bolzano, Paradoxa of the Infinite, 1851 and H. Dingler, Philosophy of Logics and Arithmetic, 1931). People say, that, if such calculating were possible, then it would have to be true according to the formula for the summation of the geometrical series that $\sum_{n=0}^{\infty} 2^n = -1$, an equation, which, of course does not make sense and should properly be replaced by our equation:

$$\sum_{n=0}^{N_\Omega} 2^n = -1 + 2^{N_\Omega+1}$$

which contains an infinite additional member.

More interesting is the behavior of the series in the infinitesimal vicinity of the limit of convergence. We perform the inessential substitution $x \to -x$ and obtain:

$$\sum_{n=0}^{N_\Omega} (-1)^n x^n = \frac{1 - (-1)^{N_\Omega+1} x^{N_\Omega+1}}{1 + x}.$$

Let us now consider $x = 1 - y\,\tilde{\omega}\left(\tilde{\omega} = \frac{1}{N_\Omega}\right)$ for rational y. We then get:

$$\sum_{n=0}^{N_\Omega} (-1)^n x^n \cong \frac{1 - (-1)^{N_\Omega+1} e(-y)}{2},$$

such that already for this y, which is contained in the unity-circle, the

series differs from the number $\frac{1}{2} \cong \frac{1}{1+x}$ by a finite margin. The value 1/2 (which the Abel's summation determines as the sum-value for the series for x = 1) is only obtained for such $x = 1 - y\widetilde{\omega}$ out of the class of the real numbers, for which y is infinitely large, for instance for

$y = \log N_\Omega$.
$(N_\Omega^2)$

Again with our specifically chosen series, we now want to investigate, what kind of interpretation we have to give to the method of the $C_1$-summation. The $n^{th}$ partial sum is:

$$s_n = \frac{1-(-1)^{n+1}x^{n+1}}{1+x},$$

and the $C_1$-sum comes about as the limit for $n = \Omega - 1$ of the arithmetic means of the $n+1^{st}$ partial sums, the result being:

$$S = \frac{1}{\Omega}\sum_0^{\Omega-1} S_n = \frac{1}{1+x} + \frac{x}{\Omega(x+1)}\left\{\frac{1+(-1)^{\Omega+1}x^\Omega}{1+x}\right\}.$$

Responsible for the deviation from 1/1 + x is the expression:

$$\omega\frac{x}{(1+x)^2}\left(1-(-1)^\Omega x^\Omega\right).$$

This error is infinitesimal, if $|x^\Omega|$ is infinitesimal, that is surely also for $x = 1 - y\omega^r$ (y rational, $r \geqq 1$). For these $x_\Omega$ it is true therefore:

$$S \cong \frac{1}{1+x},$$

such that the limiting value of the sequence of the means of the partial sums in these cases deviates from the limiting value of the series itself in such a way as to attain a representation of the function 1/1 + x even closer to the limit of convergence. This becomes even clearer, if S itself is written as an infinite series:

$$S = \frac{1}{\Omega}\sum_{n=0}^{\Omega-1}\sum_{m=0}^{n}(-1)^m x^m = \sum_{n=0}^{\Omega-1} x^n(-1)^n\omega(\Omega-n).$$

To the original series we have added the "convergence-producing" series:

$$\sum_{n=0}^{\Omega-1}(-1)^{n+1}n\omega x^n$$

the members of which are infinitesimally small for finite n, a series, however, which in total delivers a finite value.

## Par. 4 Conditionally Convergent Series

We have seen already, that in this extension some of the difficulties

of the conventional analysis can be avoided owing to the interchangeability of the order in the limiting processes. A similar fact is also true for the rearranging and the multiplication of series, which in conventional analysis only converge conditionally.

Let us consider as a first example the Leibnitz's series for log 2. It should be remarked, that the following relation holds true:

$$\sum_1^{\Omega} \frac{1}{n} - \log \Omega \cong C$$

where the logarithm is taken in relation to a sufficiently large $\Omega$ -number, for instance $\Omega^{\Omega}$: log stands for $\log_{(\Omega^{\Omega})}$. C is a real number from a class, which we denote as Euler's constant. The proof can be taken over from conventional analysis. Therefore, it is true up to an infinitesimal error:

$$\sum_1^{\Omega} \frac{1}{2n} = \frac{1}{2} \sum_1^{\Omega} \frac{1}{n} \cong \frac{1}{2} \log \Omega + \frac{C}{2}$$

and likewise

$$\sum_1^{\Omega} \frac{1}{2n+1} = \sum_1^{2\Omega+1} \frac{1}{n} - \sum_1^{\Omega} \frac{1}{2n} \cong \log(2\Omega+1) - \frac{1}{2} \log \Omega + \frac{C}{2},$$

and hereafter, again up to an infinitesimal error:

$$\sum_1^{2\Omega+1} \frac{(-1)^{n+1}}{n} = \sum_1^{\Omega} \frac{1}{2n+1} - \sum_1^{\Omega} \frac{1}{2n} \cong \log(2\Omega+1) - \log \Omega$$
$$\cong \log(2+\omega) \cong \log 2.$$

As is well known, in conventional analysis one goes on to show, that the series:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots$$

which arises from Leibnitz's by rearranging, has the sum 3/2 log 2. This we have to interpret in our case. Since Leibnitz's series has a real sum, we may according to the principal theorem for classes of real numbers extend the summation up to $4\Omega - 1$ instead of $\Omega$ and we may write:

$$\log 2 \cong \sum_1^{4\Omega-1} \frac{(-1)^{n+1}}{n} = \sum_1^{\Omega} \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) - \sum_{\Omega+1}^{2\Omega-1} \frac{1}{2n}$$
$$\cong \sum_1^{\Omega} \left( \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} \right) - \frac{1}{2} \log 2 \quad [10])$$

For the series which in the conventional analysis we get 3/2 log 2 as the sum of the rearranged series, we obtain the same result here. But in our sense the new series is not a rearrangement of the Leibnitz's series, because it lacks infinitely many (negative) members, which deliver a sum, which just amounts to the difference of the two series.

Generally it is quite easy to see, that the sum of a series should be independent of the order of its members, as soon as care is taken that

the rearranged series contains all (and only those) members of the original series. This manner of speaking is meant to include here that the statement also holds true for the components in the corresponding way. Thus in principle the difference between absolute and conditional convergence is disposed of here. The difference remains important, however, for the practical application insofar as arbitrary rearrangements are allowed for absolutely convergent series without added considerations.

Corresponding considerations can be given for the multiplication of series. Given two series:

$$A = \sum_{0}^{\Omega} a_n \quad \text{und} \quad B = \sum_{0}^{\Omega} b_n$$

For both we choose the same upper boundary as a matter of practicality (we write $\Omega$ for it). This is according to the principal theorem without influence on the class of real numbers of the final result, if we have series on our hand, which converge in the usual sense. If we now multiply member by member (which is permitted, because member by member multiplication is allowed for the components, which are finite sums), we obtain:

$$A \cdot B = \sum_{n=0}^{\Omega} \sum_{m=0}^{n} a_m b_{n-m} + \sum_{n=0}^{\Omega-1} \sum_{m=0}^{n} a_{\Omega-n+m} b_{\Omega-m}.$$

The first double-sum is the customary Cauchy's product, the second is the error term. If at least one of the two series, say A, is absolutely convergent, then the error becomes infinitely small provided B is a real $\Omega$-number. (which can be shown in the usual way.) If both A and B are conditionally convergent, the error-sum usually does not vanish, but then according to our integral definition it can be rewritten as an integral. Put, for instance:

$$a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$$

Then after an extended calculation one obtains for the additional sum the value $\pm \pi/2$ (as a class of real numbers), if as an upper limit $2\Omega$ or $2\Omega + 1$ is chosen.

(Footnote 10 on page 38 of the original $\sum_{\Omega+1}^{2\Omega-1} \frac{1}{n} \cong \log 2$ follows from the integral-definition of the logarithm.

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